THE
MATHEMATICAL GAZETTE

EDITED BY
T. A. A. BROADBENT, M.A.
WILLS HALL, BRISTOL, 9.

LONDON
G. BELL AND SONS, LTD., PORTUGAL STREET, KINGSWAY

VOL. XXV. JULY, 1941. No. 265

THE LIBRARY.

IN view of the reduction of other facilities by the war, the maintenance of the Library service seemed so desirable that the risks of loss though they could not be ignored must be taken. To reduce them, each volume issued since the outbreak has had a jacket bearing the name of the Association and an address, and a request, to which few borrowers pay any attention, for a monthly notification if the volume is being retained.

The Librarian has moved and is unable any longer to accommodate the books in his house, and the hopeful plans for their eventual transference to the University Library in London are now in abeyance. In this emergency the Douai School has generously given the Association for the duration of the war the use of half of the shelves in a handsome new library building, together with storage for books that cannot be displayed, and the Librarian is maintaining service to members partly from his own collection and partly by occasional journeys to Woolhampton ; considerable delays are inevitable, but there is very little reduction in the material available for loan. For the first six months of the war the demand for books was small, but there was a gradual recovery and the use made of the Library has become normal.

The annual grant to the Library has never provided for overtaking arrears of binding, and for a long time it has been evident that a capital expenditure of the order of £500 would be necessary before the several hundreds of volumes of journals and some scores of books which were unbound or shabby when acquired could be accepted in any public building. A comprehensive programme of binding and repairs must form part of any plan for the ultimate care of the Library, and such a programme was drawn up in 1939. By a decision of the Council, this programme is being carried out in spite of the war, though more and more slowly as the binder's staff is depleted ; the most important runs of periodicals are now ready, but a vast amount of unspectacular work has still to be done. There is no doubt that whatever its ultimate destination the displayed Library will be a credit to the Association.

DETERMINANT EXPANSIONS.

By L. M. Milne-Thomson.

A set of numbers a_{ij} (called *elements*) arranged in a rectangular array of m rows and n columns constitutes a matrix of orders $m \times n$. For example, of the arrays

$$(1) \qquad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}, \quad \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

the first is a matrix of two rows and three columns, *i.e.* of orders 2×3, the second is a square matrix of orders 3×3.

From a square matrix of orders $n \times n$ we can form a *determinant* of order n. Thus from the above square matrix we can form the determinant of order 3

$$(2) \qquad \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \Sigma \pm a_{1\alpha} a_{2\beta} a_{3\gamma},$$

where the summation is extended to all permutations $(\alpha\beta\gamma)$ of the numbers 1, 2, 3, and the positive or negative sign is to be taken before a term of the sum according as the permutation $(\alpha\beta\gamma)$ is of the even or the odd class.

A permutation is said to be of the even or odd class according as the number of inversions of the natural order 1, 2, ..., in which a larger number precedes a smaller is even or odd. For example, the permutation (132) of the integers 1, 2, 3 contains one inversion, 3 before 2, and is therefore of the odd class. The permutation (312) contains two inversions, namely, 3 before 1, 3 before 2, and is therefore of the even class. The number of inversions (and therefore the sign of the corresponding term in a determinant) can most readily be obtained by Aitken's graphical method as follows:

The natural and proposed orders are written one above the other and corresponding numbers are joined by lines. Each inversion gives rise to one intersection of these lines, and an even number of intersections therefore indicates an even number of inversions. The method here illustrated for three consecutive integers 1, 2, 3 of course applies unmodified to any number of consecutive integers and their permutations.

Every determinant has a *main diagonal* which runs from the top left-hand corner to the bottom right-hand corner. In the present notation in the case of a determinant Δ_n of order n the term arising

from the elements of the main diagonal will be

$$+a_{11}a_{22}a_{33}\cdots a_{nn}.$$

If in a given matrix we delete all the elements of certain rows and all the elements of certain columns in such a way that the elements which remain undeleted form an array of r rows and r columns, the determinant which arises from this array is called a *minor determinant* (or simply a minor) of order r of the matrix. For example, from the first matrix of (1) we can form the three minors

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}, \quad \begin{vmatrix} a_{12} & a_{13} \\ a_{22} & a_{23} \end{vmatrix}$$

each of the second order, and the six minors

$$|a_{11}|, \quad |a_{12}|, \quad |a_{13}|, \quad |a_{21}|, \quad |a_{22}|, \quad |a_{23}|$$

of the first order. These latter are of course simply the elements of the matrix.

Considering the second matrix of (1) we can (by making no deletions) form one minor of the third order, namely, the determinant (2).

For the sake of uniformity it is useful to consider also the minor of order zero which contains no elements. To this minor we ascribe the value unity.

In what follows minors of zero order will always be considered allowable.

By a minor of order r of a given determinant we simply mean the minor obtained in the above manner from the array which consists of the elements of the determinant, the matrix of the determinant.

Now consider the determinant Δ_n of order n whose elements are a_{ij} where the row number i ranges from 1 to n and the column number j also ranges from 1 to n. If a set of minors of Δ_n, such that their row numbers consist of the numbers 1 to n without repetition and their column number consist of the numbers 1 to n without repetition, be taken as the factors of a product to which is prefixed the sign of that term of Δ_n whose elements are the main diagonal elements of the minors, this product is identical with the sum of a certain number of terms of Δ_n. Thus if $n=3$, we may consider the product

$$-\begin{vmatrix} a_{11} & a_{12} \\ a_{31} & a_{32} \end{vmatrix} \times |a_{23}|,$$

since $a_{11}a_{32}a_{23}$ has the negative sign in the expansion of Δ_3. This product is equal to

$$-a_{11}a_{32}a_{23}+a_{31}a_{12}a_{23},$$

both of which are terms of Δ_3. Observe that this process gives only terms of Δ_3 without repetition, but not necessarily all the terms.

Any two minors which are susceptible of entering into a set of the type just described are said to be *conjunctive in position* or simply conjunctive.

The whole set of minors form a set of *perjunctive* minors or simply a *perjunct*.

The product of the minors of a perjunctive set with the proper sign prefixed is called a *signed perjunct*.

To illustrate these ideas consider

$$\Delta_5 = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} \end{vmatrix}.$$

Any two of the minors

$$1, \quad |a_{11}|, \quad \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{44} & a_{45} \\ a_{54} & a_{55} \end{vmatrix}$$

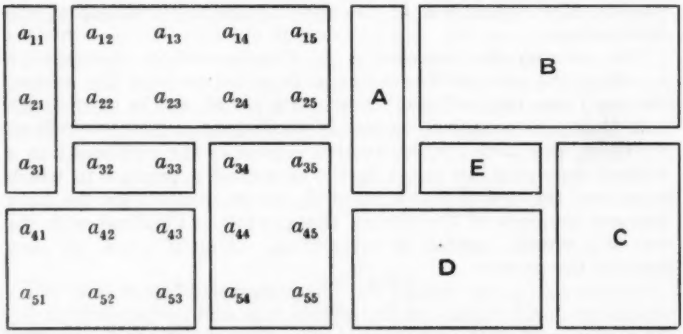
are conjunctive. The four form a perjunct. Their product with the positive sign is a signed perjunct.

Again the product of the conjunctive minors

$$\begin{vmatrix} a_{12} & a_{14} \\ a_{22} & a_{24} \end{vmatrix}, \quad \begin{vmatrix} a_{31} & a_{33} & a_{35} \\ a_{41} & a_{43} & a_{45} \\ a_{51} & a_{53} & a_{55} \end{vmatrix}$$

with a minus sign prefixed yields a signed perjunct.

The matrix of a determinant can be partitioned (in various ways) so that the elements are enclosed by *panels*, that is to say, rectangles with " horizontal " and" vertical " sides. For example,



the panels of this particular partitioning being shown separately on the right with the letters *ABCDE*. It is essential only that every element should be in one panel, and that no two panels should enclose the same element.

We now prove an expansion theorem due to Rice.*

* L. H. Rice (*American Jour. of Math.* 42, p. 237 (1920)). This remarkable and general theorem appears to have been overlooked by writers on determinants.

Rice's theorem.

If the matrix of a determinant Δ be partitioned into panels in any manner, then Δ can be expanded as the sum of all signed perjuncts composed of one minor from each panel.

Let there be k panels denoted by $A, B, C, \ldots, F$. Consider any term of Δ. Separate its elements into groups consisting of those elements which lie in panel A, those which lie in panel B, and so on. These groups determine by their row and column numbers k minors (some of which may contain no elements) whose elements lie on the panels $A, B, \ldots$ respectively. These minors form a perjunct which is the only perjunct of k minors belonging one to each of the panels and which contains the given term of Δ.

Thus the sum of the signed perjuncts specified in the theorem contains nothing but terms of Δ and contains every term once and once only. It is therefore an expansion of Δ. Q.E.D.

As a simple illustration of the expansion consider

$$\Delta_4 = \left|\begin{array}{c:ccc} a_{11} & a_{12} & a_{13} & a_{14} \\ \hdashline a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{array}\right|$$

where the dotted lines show the method of partitioning into 4 panels. Rice's theorem then gives the expansion

$$a_{11}\begin{vmatrix} a_{22} & a_{23} & a_{24} \\ a_{32} & a_{33} & a_{34} \\ a_{42} & a_{43} & a_{44} \end{vmatrix}$$

$$-a_{12}a_{21}\begin{vmatrix} a_{33} & a_{34} \\ a_{43} & a_{44} \end{vmatrix} + a_{12}a_{31}\begin{vmatrix} a_{23} & a_{24} \\ a_{43} & a_{44} \end{vmatrix} - a_{12}a_{41}\begin{vmatrix} a_{23} & a_{24} \\ a_{33} & a_{34} \end{vmatrix}$$

$$+a_{13}a_{21}\begin{vmatrix} a_{32} & a_{34} \\ a_{42} & a_{44} \end{vmatrix} - a_{13}a_{31}\begin{vmatrix} a_{22} & a_{24} \\ a_{42} & a_{44} \end{vmatrix} + a_{13}a_{41}\begin{vmatrix} a_{22} & a_{24} \\ a_{32} & a_{34} \end{vmatrix}$$

$$-a_{14}a_{21}\begin{vmatrix} a_{32} & a_{33} \\ a_{42} & a_{43} \end{vmatrix} + a_{14}a_{31}\begin{vmatrix} a_{22} & a_{23} \\ a_{42} & a_{43} \end{vmatrix} - a_{14}a_{41}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

The expansion derived from the above particular method of partitioning is known as Cauchy's expansion.

Observe that the first perjunct contains two minors of zero order and each other perjunct one such minor, since the statement of Rice's theorem demands a minor from each panel.

By suitable disposition of the panels Rice's theorem contains a simultaneous justification for all special expansions of a determinant. One such special method is known as Laplace's expansion, the appropriate partitioning being then performed by drawing a " horizontal " line between two rows of the matrix (or a " vertical " line between two columns). More generally it is not essential to suppose all the rows belonging to a Laplace partitioning to be con-

secutive. It is also possible to draw more than one horizontal line, thus forming three or more panels for a generalised Laplace expansion. These obvious developments are left to the reader. The well-known method of expanding by elements of the top row is a special case.

There is another theorem given by Albeggiani * which is of two-fold interest, for it affords a second proof of Rice's theorem and also leads to a simple and direct proof of the theorem for multiplication of determinants. In order to appreciate the enunciation a few preliminary remarks are required.

Consider the general determinant $\Delta = |\, a_{1n} \,|$ and put

$$a_{ij} = h_{ij}^{(1)} + h_{ij}^{(2)} + \ldots + h_{ij}^{(r)},$$

so that each constituent is a polynomial of r terms any or all of which may be zero.

Set up the r determinants

$$\Delta^{(k)} = \begin{vmatrix} h_{11}^{(k)} & \ldots & h_{1n}^{(k)} \\ \cdot & \ldots & \cdot \\ h_{n1}^{(k)} & \ldots & h_{nn}^{(k)} \end{vmatrix} \qquad k = 1, 2, \ldots, r.$$

Form a *mixed perjunct* by taking one minor from $\Delta^{(1)}$, a second minor conjunctive in position from $\Delta^{(2)}$, and so on to $\Delta^{(r)}$, and prefix the sign determined precisely as it would be determined if all the minors came from one determinant. Then we have *Albeggiani's theorem* as follows :

If Δ be any determinant, the sum of all the signed mixed perjuncts from r determinants so formed that the sum of their matrices is the matrix of Δ, is an expansion of Δ.

Proof. Consider any term of Δ. This a-term yields r^n monomials each the product of n of the h's which may be called h-terms of Δ.

Now obviously we can think of expanding Δ directly into its h-terms without first forming the a-terms.

From that point of view it is clear that any given h-term can be found in one and only one mixed perjunct.

For separate the elements of this h-term into the $h^{(1)}$'s, the $h^{(2)}$'s, and so on.

These groups determine just one mixed perjunct containing this term, and therefore as the perjuncts contain nothing but h-terms of Δ we have an expansion of Δ.

Q.E.D.

Second proof of Rice's theorem.

Proof. By means of Albeggiani's theorem we form $\Delta^{(1)}$, $\Delta^{(2)}$, ..., $\Delta^{(r)}$ by writing into r blank matrices the r panels of Δ, each in its proper place, and then filling up each matrix with zeros.

All minors of $\Delta^{(1)}$ vanish except those lying in the first panel,

* M. Albeggiani (*Giornale di Mat.*, 13, p. 1 (1874)). The proof which follows is due to Rice, *loc. cit.*

all minors of $\Delta^{(2)}$ vanish except those lying in the second panel, and so on.

The mixed perjuncts which survive are identical with the perjuncts of Δ specified in the enunciation. Q.E.D.

We can use Albeggiani's theorem to obtain the rule for multiplying determinants. We show the method, which is perfectly general, for the special case of determinants of the third order.

Consider the product

$$P = \begin{vmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \beta_1 & \beta_2 & \beta_3 \\ \gamma_1 & \gamma_2 & \gamma_3 \end{vmatrix} \times \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= \Sigma \pm \alpha_\lambda \beta_\mu \gamma_\nu a_p b_q c_r,$$

where $(\lambda\mu\nu)$, (pqr) are each permutations of 1, 2, 3 and the plus or minus sign is prefixed according as these permutations are of the same or different classes. It is then clear that in each term of the sum the Greek and Latin suffixes occur in pairs. Thus P is the sum of all signed mixed perjuncts obtained by taking a minor of order 1 (*i.e.* an element) from each of the matrices

$$\Delta_1 = \begin{bmatrix} \alpha_1 a_1 & \alpha_1 b_1 & \alpha_1 c_1 \\ \beta_1 a_1 & \beta_1 b_1 & \beta_1 c_1 \\ \gamma_1 a_1 & \gamma_1 b_1 & \gamma_1 c_1 \end{bmatrix}$$

$$\Delta_2 = \begin{bmatrix} \alpha_2 a_2 & \alpha_2 b_2 & \alpha_2 c_2 \\ \beta_2 a_2 & \beta_2 b_2 & \beta_2 c_2 \\ \gamma_2 a_2 & \gamma_2 b_2 & \gamma_2 c_2 \end{bmatrix}$$

$$\Delta_3 = \begin{bmatrix} \alpha_3 a_3 & \alpha_3 b_3 & \alpha_3 c_3 \\ \beta_3 a_3 & \beta_3 b_3 & \beta_3 c_3 \\ \gamma_3 a_3 & \gamma_3 c_3 & \gamma_3 c_3 \end{bmatrix}$$

Moreover every minor of an order higher than 1 in each of these matrices vanishes. Hence by Albeggiani's theorem :

$$P = \begin{vmatrix} \alpha_1 a_1 + \alpha_2 a_2 + \alpha_3 a_3 & \alpha_1 b_1 + \alpha_2 b_2 + \alpha_3 b_3 & \alpha_1 c_1 + \alpha_2 c_2 + \alpha_3 c_3 \\ \beta_1 a_1 + \beta_2 a_2 + \beta_3 a_3 & \beta_1 b_1 + \beta_2 b_2 + \beta_3 b_3 & \beta_1 c_1 + \beta_2 c_2 + \beta_3 c_3 \\ \gamma_1 a_1 + \gamma_2 a_2 + \gamma_3 a_3 & \gamma_1 b_1 + \gamma_2 b_2 + \gamma_3 b_3 & \gamma_1 c_1 + \gamma_2 c_2 + \gamma_3 c_3 \end{vmatrix}$$

which is the usual row × column rule which holds also for matrices.

In conclusion we may observe that most of the elementary theorems for manipulation of determinants, addition of rows, vanishing of a determinant with two identical rows, and so on, follow at once from Rice's theorem. The considerations here given might therefore form the basis of an introductory treatment of the subject. L. M. M.-T.

GLEANINGS: AN APPEAL.

The Editor will be grateful for help in the filling up of odd corners. A precise reference should accompany every quotation.

ON A PROBLEM IN ELEMENTARY GEOMETRY.

By F. G. Maunsell and D. Pedoe.

The rectangle $ABCD$ is supposed to lie in a vertical plane, with AB horizontal. P is any point in the horizontal plane through CD. Prove that $\angle APB < \angle CPD$.

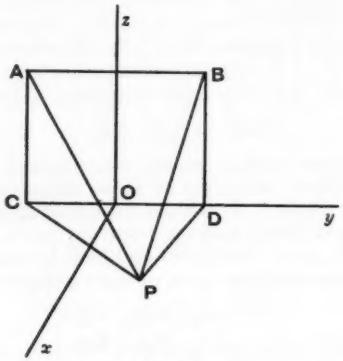


Fig. 1.

Provided that the perpendicular from P on to CD meets CD between C and D, this theorem is true and easily proved by comparing the angles into which a plane through P perpendicular to the rectangle divides the two angles considered. The sine of each of the angles into which $\angle APB$ is divided is less than the sine of the corresponding angle in $\angle CPD$.

This proof breaks down when the perpendicular from P on to CD does not satisfy the conditions. The main question discussed in this note is "Are there points P for which $\angle APB \geqslant \angle CPD$, and if so, where are they?" In a correspondence between the authors, of whom one is at present in the army, a result was first obtained geometrically. Although this is longer than the algebraic proof subsequently produced, it is given for its possible interest. The algebraic proof follows, and an application of the main result is made to a connected problem, a theorem on the orthogonal projection of an angle being obtained.

1. If P is a point for which $\angle APB = \angle CPD$, the perpendicular from P on to CD cuts it at a point outside CD. Suppose the point is nearer C than D. Since the angles APB, CPD are equal, and $AB = CD$, the circumradius of APB equals that of CPD.

If we project the circumcircle of APB orthogonally on to the horizontal plane through CD, we obtain an ellipse circumscribing CDP, CD being parallel to the major axis, and the circumradius of CDP will equal the semi-major axis of the ellipse in length. (Fig. 2.)

2. We now consider the following problem: "C, D are points with respective parameters α, $\pi-\alpha$ on the ellipse $a\cos\theta$, $b\sin\theta$. P is the point with parameter $\pi+\phi$, where $\sin\phi<\sin\alpha$. To determine ϕ, given that the circumradius of $CDP=a$."

The coordinates of the centre of the circle are (ξ, η) where

$$\xi=0, \quad \eta=(a^2-b^2)(\sin\phi-\sin\alpha)/2b.$$

(cf. Smith's *Conics*, p. 169.)

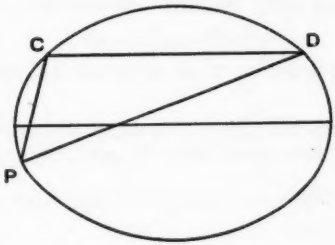


FIG. 2.

Hence

$$a^2=a^2\cos^2\alpha+[b\sin\alpha-(a^2-b^2)(\sin\phi-\sin\alpha)/2b]^2.$$

This becomes

$$(a^2-b^2)\sin\phi=(a\pm b)^2\sin\alpha.$$

Thus $$\frac{\sin\phi}{\sin\alpha}=\frac{a\pm b}{a\mp b}=\frac{a-b}{a+b}, \text{ since } \sin\phi<\sin\alpha.$$

Returning to the original problem, let ψ be the angle between the planes APB, CPD. Then $b=a\cos\psi$, and

(1) $$\sin\phi/\sin\alpha=\tan^2\tfrac{1}{2}\psi;$$

i.e. the ratio of the distances of P and CD from the major axis of the ellipse is equal to $\tan^2\frac{1}{2}\psi:1$.

Now consider a line parallel to CD in the horizontal plane through CD. We find the points on it at which AB, CD subtend equal angles. Let

$$AB=CD=2d, \ CA=h,$$

and choose rectangular axes as shown in the first figure, with the origin at the mid-point of CD. Let the equation of our line be

$$x=h\cot\psi, \ z=0.$$

Divide the join of the points

$$F\equiv(h\cot\psi, 0, 0); \ E\equiv(0, 0, h)$$

in the ratio $\tan^2\frac{1}{2}\psi:1$. We obtain a point G with coordinates

$$(h\cot\psi\cos^2\tfrac{1}{2}\psi, 0, h\sin^2\tfrac{1}{2}\psi).$$

With centre G and radius GA describe a sphere. The two points of intersection of this sphere with our line give the required points.

The proof follows simply from the remarks that the diameter of the circumcircle of ABP which is parallel to AB projects into the major axis of the ellipse around CDP, and that the ratio $FG : GE$ is unaltered by orthogonal projection.

Straightforward algebra gives the coordinates of the points on the line as

$$(h \cot \psi, y, 0), \text{ where } y^2 = d^2 + x^2 \sec \psi, \text{ i.e.}$$

$$y^2 = d^2 + x\sqrt{(x^2 + h^2)} \tag{2}$$

is the locus of the points in $z = 0$ at which AB, CD subtend equal angles.

3. We now consider a more direct proof of this result. With the axes chosen, let P have coordinates $(x, y, 0)$. We may assume that $y > d$. Then

$$\angle CPD = \tan^{-1}\{(y+d)/x\} - \tan^{-1}\{(y-d)/x\}.$$

Thence $$\tan CPD = 2dx/(x^2 + y^2 - d^2),$$

$$\angle APB = \tan^{-1}\{(y+d)/x'\} - \tan^{-1}\{(y-d)/x'\}.$$

Thus $$\tan APB = 2dx'/(x'^2 + y^2 - d^2),$$

where $$x'^2 = x^2 + h^2.$$

These angles are equal when

$$\frac{x}{x^2 + y^2 - d^2} = \frac{\sqrt{(x^2 + h^2)}}{x^2 + h^2 + y^2 - d^2}$$

or $$x^2(x^2 + h^2 + y^2 - d^2)^2 = (x^2 + h^2)(x^2 + y^2 - d^2)^2,$$

i.e. $$2x^2(x^2 + y^2 - d^2) + x^2h^2 = (x^2 + y^2 - d^2)^2,$$

or $$x^2h^2 + x^4 = [x^2 + y^2 - d^2 - x^2]^2 = (y^2 - d^2)^2.$$

Thus $y^2 = d^2 + x\sqrt{(x^2 + h^2)}$, as before.

If we consider the function

$$f(x, y, d) = \frac{2d\sqrt{(x^2 + h^2)}}{x^2 + h^2 + y^2 - d^2} - \frac{2dx}{x^2 + y^2 - d^2},$$

reduced to a common denominator, we see that the coefficient of y^2 in the numerator is positive. The denominator is positive while $y^2 > d^2$, and therefore, if x is kept constant, $f(x, y, d) > 0$ as soon as $y^2 > d^2 + x\sqrt{(x^2 + h^2)}$. This means that for such values of y, $\angle APB > \angle CPD$.

4. In order to apply this result we slightly change our point of view. The angle APB subtended at $(x, y, 0)$ by AB is equal to the angle CQD subtended by CD at the point Q with coordinates (x, y, h). We therefore say "If C, D are the points $(0, \pm d, 0)$, if P is $(x, y, 0)$, and Q is (x, y, z), then $\angle CPD < \angle CQD$ if

$$y^2 > d^2 + x\sqrt{(x^2 + z^2)}." \tag{3}$$

We now obtain a theorem which gives a sufficient condition for the orthogonal projection of an angle to be less than the original angle. Let the angle $CQD = \alpha$, and let the acute angle made by the internal bisector of $\angle CQD$ with CD be γ. Then we find, without much difficulty, that the coordinates of Q, referred to our axes, can be written as

$$(x, y, z) \equiv [d(\cos\alpha - \cos 2\gamma)\cos\psi \operatorname{cosec}\alpha,\ d\sin 2\gamma \operatorname{cosec}\alpha,$$
$$d(\cos\alpha - \cos 2\gamma)\sin\psi \operatorname{cosec}\alpha].$$

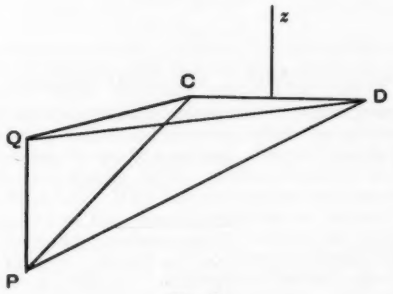


FIG. 3.

Hence the orthogonal projection of $\angle CQD$, *i.e.* $\angle CPD < \angle CQD$ if

$$x^2z^2 < (y^2 - d^2)^2 - x^4,$$

i.e. $(\cos\alpha - \cos 2\gamma)^4 \sin^2\psi \operatorname{cosec}^4\alpha \cos^2\psi$
$$< (\sin^2 2\gamma \operatorname{cosec}^2\alpha - 1)^2 - (\cos\alpha - \cos 2\gamma)^4 \cos^4\psi \operatorname{cosec}^4\alpha,$$

i.e. $(\cos\alpha - \cos 2\gamma)^4 \operatorname{cosec}^4\alpha \cos^2\psi < (\sin^2 2\gamma \operatorname{cosec}^2\alpha - 1)^2$

$$(\cos\alpha - \cos 2\gamma)^2 \operatorname{cosec}^2\alpha \cos\psi < (\sin^2 2\gamma \operatorname{cosec}^2\alpha - 1)$$
$$(\text{since } 2\gamma > \alpha),$$

i.e. $$(\cos\alpha - \cos 2\gamma)\cos\psi < \frac{\sin^2 2\gamma - \sin^2\alpha}{\cos\alpha - \cos 2\gamma}$$
$$= \cos\alpha + \cos 2\gamma,$$

i.e. if $\cos\alpha(1 - \cos\psi) + \cos 2\gamma(1 + \cos\psi) > 0$.

The condition is therefore

$$\cos 2\gamma + \tan^2 \tfrac{1}{2}\psi \cos\alpha > 0. \tag{4}$$

This condition is certainly satisfied if $\cos 2\gamma > 0$, *i.e.* if $\gamma < \frac{1}{4}\pi$, for this entails the condition $\alpha < \frac{1}{2}\pi$. We therefore have the theorem :

" A sufficient condition for the orthogonal projection of an angle CQD to be less than the original angle is that the internal bisector of CQD make an angle $< \frac{1}{4}\pi$ with the axis of projection CD."

Comparing the condition (4) with the equation (1) suggests that in any triangle CQD the ratio of the distances of Q and CD from the diameter of the circumcircle which is parallel to CD is equal to

$$-\cos 2\gamma : \cos \alpha,$$

where $\angle CQD = \alpha$, and γ is the acute angle made by the internal bisector of $\angle CQD$ with CD.

This can be verified directly, and suggests an alternative proof of the condition (4), based on the geometric argument first given.

F. G M.
D. P.

GLEANINGS FAR AND NEAR.

1361. But we must return to the general question of the educational treatment of mathematics, on which I have still a word to say. I take my text this time from Professor Chrystal. Speaking of algebra, he says: "The whole training consists in example-grinding. What should have been merely the help to attain the end has become the end itself. The result is that algebra, as we teach it, is neither an art nor a science, but an ill-digested farrago of rules, whose object is the solution of examination problems." This is strong language from the writer of a great book on algebra: perhaps it is too strong. In schools, at any rate, it must be allowed that the solution of problems affords an admirable, though one-sided, mental training. Yet, it is far from an objection to a study that its educational efficacy is restricted and specific, provided we do not expect more of it than it can do. I am told, however, that this example-grinding must be persevered in, in order that the student may attain due facility in the use of notations and devices. This line of defence seems to point to a defect. To the boy who goes right through this grind the drill may do some good, and such a one is likely to be a mathematical specialist. But what to the boy who does not go through with it, whose speciality is not mathematics? It is as if, having only the means to build myself a cottage, I expend it all in building what would be an excellent porch to a castle; or, having only half-an-hour in which to make a sketch, I give all the time to a five-barred gate in the foreground. If intentional, it looks like the sacrifice of proportion to a mistaken idea of thoroughness; but it is probably not intentional, it is probably the unforeseen outcome of circumstances. Mathematical text-books have been usually written by mathematicians, not by educationists; and they have usually mathematics and not education as their shaping idea—in a word, the writers are usually professionals, and their aim is primarily professional. Now, the question I would ask is this: Would not the general education value of mathematics be increased if some of the pains taken to ensure expertness of manipulation were directed to ensuring rather more insight, and something of a general survey of mathematics as a system? It is surely often better to have an outline-map of a whole country rather than a piece of the Ordnance Survey of the same size as the map. This, I take it, is what Professor Chrystal is fighting for when he denounces the practice of "retarding their (the students') progress by making the details and illustrations of particular rules and methods ends in themselves".—James Ward, *Psychology applied to Education.*

The material of Ward's book was given as a course of lectures to teachers in 1880. The book was edited by Professor G. Dawes Hicks and published in 1926. [Per Mr. H. Lowery.]

MOTION OF A BODY WHOSE MASS IS CHANGING.

By A. S. Ramsey.

In Routh's *Elementary Rigid Dynamics* (Sixth edition, 1897, § 300) a section is given to the application of the principle of linear momentum to determine the gradual changes produced by alterations of mass, with examples ; but I cannot recall that any modern textbook explicitly points out that the formulae

$$M\,dV/dt = F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1)$$

and

$$d(MV)/dt = F, \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2)$$

which serve to determine the motion of a body of constant mass, are not necessarily true when the mass is changing.

In Besant's *Dynamics* (Fifth edition, 1912, p. 43) we read : "Even if the mass which is acted upon be variable, the effect of force upon it is the production of momentum and the measure of the force is the rate of that production, so that

$$P = d(mv)/dt,$$

or, in other words, force is the time-flux of the momentum." And in the following chapter this formula is applied to the solution of several problems involving a changing mass, such as the fall of a raindrop and the fall of snow down a sloping roof, and it undoubtedly leads to correct results in all the cases considered by the author.

There is of course no question as to generality of the Newtonian measure of force as rate of change of momentum ; but it is a fact that there are cases in which this rate of change cannot be completely represented by a single differential coefficient as in (1) or (2) but contains an additional term. We shall show that it is easy to make examples about the motion of a variable mass in which (1) is true and (2) is false, or *vice versa*, as well as examples in which neither form is correct ; though in every case the correct measurement of the impressed force is the rate of change of momentum which it produces.

Examples. (i) *An engine of total mass M at time t is moving with velocity V along straight rails and water is leaking vertically from it at the rate of m units of mass per unit time. The net propelling force is F. Find the equation of motion.*

The method we adopt in all such examples is to state the momentum at time t and the momentum of the same mass at time $t+\delta t$ and equate the increment in the momentum to $F\,\delta t$. Then divide by δt and make $\delta t \to 0$.

In this case,

at time t momentum $= MV$,

and at time $t+\delta t$ ..... $= (M - m\,\delta t)(V + \delta V) + m\,\delta t(V + \epsilon\,\delta V)$,

where $0 < \epsilon < 1$; the last term representing the momentum of the

water which leaks in time δt, whose average velocity lying between V and $V+\delta V$ may be denoted by $V+\epsilon\,\delta V$.

Hence $(M-m\,\delta t)(V+\delta V)+m\,\delta t(V+\epsilon\,\delta V)-MV=F\,\delta t$,

and this leads to $M\,dV/dt=F$. .....(3)

In this case, therefore, formula (1) gives the equation of motion, although M is variable and the result is apparently independent of m.

(ii) *The same problem, but instead of dropping water let the engine pick up water at the rate m from a stream flowing between the rails with velocity v in the direction of motion of the engine.*

Here the momentum at time t of the engine and the mass of water to be picked up in time $\delta t = MV+m\,\delta t\,.\,v$.

At time $t+\delta t$ the momentum of the same mass

$$=(M+m\,\delta t)(V+\delta V).$$

Hence $(M+m\,\delta t)(V+\delta V)-MV-m\,\delta t\,.\,v=F\,\delta t$,

leading to $M\,dV/dt+mV-mv=F$.

Since m is the rate of increase of M, i.e. dM/dt, therefore the result is $d(MV)/dt-mv=F$. .....(4)

It follows that in the special case in which the water is picked up from rest, i.e. when $v=0$, and only in that case does formula (2) represent the motion.

(iii) *A similar problem in which, instead of losing or picking up water, we suppose that in each unit of time the engine converts a mass m of fuel into a mass m′ of smoke and gas and ejects it vertically.*

At time t the momentum $=MV$.

..... $t+\delta t$ ..... $=(M-m\,\delta t)(V+\delta V)+m'\delta t(V+\epsilon\,\delta V)$,

where $0<\epsilon<1$; the last term representing the momentum in the direction of V of the matter ejected in time δt, its average velocity on leaving lying between V and $V+\delta V$.

Hence $(M-m\,\delta t)(V+\delta V)+m'\delta t(V+\epsilon\,\delta V)-MV=F\,\delta t$,

leading to $M\,dV/dt-mV+m'V=F$.

Here m represents the rate at which M is decreasing, i.e. $m=-dM/dt$, and the result may be written

$$d(MV)/dt+m'V=F \quad .....(5).$$

It follows that only in the special case in which the products of combustion are neglected, i.e. when $m'=0$, does formula (2) represent the motion.

If we now look back and compare the results, we notice that in Ex. (i), where m is the rate of decrease of M, i.e. $m=-dM/dt$, (3) may be written

$$d(MV)/dt-V\,dM/dt=F$$

or $$d(MV)/dt + mV = F.$$

Also, since m is the rate of change of a mass, the term mV has dimensions (momentum)/(time) and the equation states that the impressed force F is the rate of change of the momentum of the engine and its contents plus the rate at which the dropping water carries off momentum.

Similarly in Ex. (ii), result (4) shows that the impressed force is equal to the rate of change of the momentum of the engine and its contents less the rate at which it is picking up momentum from the stream. And in Ex. (iii), result (5) shows that the impressed force is equal to the rate of change of the momentum of the engine and its contents plus the rate at which the products of combustion carry off momentum.

In all cases the results are consistent with the definition of force as rate of change of momentum, but the examples show that the forms (1) and (2) do not include all cases. We conclude that unless it is evident from the outset what form the equation of motion will take in a particular case, the only safe procedure is to obtain the equation by considering the difference of the momenta at the ends of an interval of time δt in the manner exemplified above. A. S. R.

1362. Well, it having been decided to bet, the next question was, how to bet? Now, Henry had read a magazine article concerning the tables at Monte Carlo, and, being of a mathematical turn, had clearly grasped the principles of the game. He said to himself, with his characteristic caution, " I'll wait till red wins four times running, and then I'll stake on the black."

(" But surely ", remarked the logical superior person in him, " you don't mean to argue that a spin of the ball is affected by the spins that have preceded it? You don't mean to argue that, because red wins four times, or forty times, running, black is any the more likely to win at the next spin? " " You shut up! " retorted the human side of him crossly. " I know all about that.")—Arnold Bennett, *A Great Man*, Ch. XXV. [Per Mr. A. F. Mackenzie.]

1363. Four oarsmen settled themselves . . . and waited submissively for the orders of young Francis Perrin. . . . Jean Perrin pulled on his oar with such force that he made the boat swirl round. Émile Borel was behind Perrin, and behind Borel, in the bow, was Marie Curie. . . . Severe but just criticisms by the young coxswain broke the silence : " Number two is slacking! " Émile Borel tried to deny his fault, but soon resigned himself and, forgetting his laziness, pulled harder at the oar.—E. Curie, *Madame Curie* (English translation, 1939), pp. 327–8.

1364. I do not wish to add anything to what Mr. Clarke has said, and the only thing that I notice that I should like to subtract is that at the top of a cyclone air passes from low pressure to high pressure, just opposite to what it does at the bottom. I know many people think so, and some have said so, but frankly I do not believe it does—the bottom is the bottom, and the top, if there is one, is the top, and not a negative bottom.—Sir Napier Shaw in the preface to G. A. Clarke, *Clouds* (Constable), 1920. [Per Mr. B. Spain.]

ON A MUCH-NEGLECTED EQUATION.

By E. G. Phillips.

In Analytical Geometry there is a most useful equation which is very rarely used and seems to be very unfamiliar to candidates in school examinations. It is the equation of the chord of a conic whose mid-point is (x_1, y_1). Speaking from a long experience as an examiner, I am always astonished at the unfamiliarity of candidates with this particular equation. If one sets a question asking the candidates to obtain this equation and follows this with a simple rider designed to show its usefulness, even if the candidates know how to find the equation, they rarely solve the rider by means of it.

The easiest way of obtaining the equation of the chord whose mid-point is $Q_1(x_1, y_1)$ is to write the equation of the chord in the form

$$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r.$$

If the conic is $S=0$, where $S\equiv ax^2+2hxy+by^2+2gx+2fy+c$, on substituting $x=x_1+r\cos\theta$, $y=y_1+r\sin\theta$ in $S=0$, we get a quadratic in r of the form

$$Ar^2+2r\{(ax_1+hy_1+g)\cos\theta+(hx_1+by_1+f)\sin\theta\}+B=0.$$

If Q_1 is the mid-point of this chord, the roots of this equation must be equal in magnitude and opposite in sign: hence the coefficient of r is zero. It follows that the equation of the chord is

$$(x-x_1)(ax_1+hy_1+g)+(y-y_1)(hx_1+by_1+f)=0,$$

or

$$(x-x_1)\frac{\partial S}{\partial x_1}+(y-y_1)\frac{\partial S}{\partial y_1}=0.$$

Another method of obtaining the equation is by using Joachimsthal's method. If $Q_1(x_1, y_1)$ and $Q_2(x_2, y_2)$ are any two points in the plane of the conic $S=0$, where $S\equiv ax^2+2hxy+by^2+2gx+2fy+c$, by expressing the condition that the point dividing Q_1Q_2 in the ratio $\lambda:1$ lies on S, we obtain Joachimsthal's equation:

$$\lambda^2S_2+2\lambda P_{12}+S_1=0, \quad \text{.....(J)}$$

where $S_1\equiv ax_1^2+2hx_1y_1+\ldots+c$, $S_2\equiv ax_2^2+2hx_2y_2+\ldots+c$,

$$P_{12}\equiv ax_1x_2+h(x_1y_2+x_2y_1)+\ldots+g(x_1+x_2)+\ldots+c.$$

If R_1, R_2, are the points at which Q_1Q_2 meets S, then the roots λ_1, λ_2 of (J) are $\lambda_1=Q_1R_1/R_1Q_2$, $\lambda_2=Q_1R_2/R_2Q_2$.

If Q_1 be the mid-point of R_1R_2, then $R_2Q_1=Q_1R_1$ and

$$\frac{1}{\lambda_1}=\frac{R_1Q_2}{Q_1R_1}=\frac{R_1R_2}{Q_1R_1}+\frac{R_2Q_2}{R_2Q_1}=-2-\frac{R_2Q_2}{Q_1R_2}=-2-\frac{1}{\lambda_2}.$$

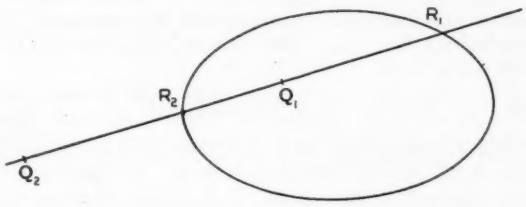
Hence the condition that Q_2 lies on the chord whose mid-point is Q_1 is $\lambda_1+\lambda_2=-2\lambda_1\lambda_2$ or $-2P_{12}/S_2=-2S_1/S_2$, and, since $S_2\neq 0$,

the condition becomes

$$P_{12} = S_1. \quad \text{.....(1)}$$

The locus of the points Q_2 is obtained by dropping the suffixes 2 in equation (1) and so the equation of the chord whose mid-point is $Q_1(x_1, y_1)$ is

$$P_1 = S_1.$$



This equation is readily thrown into the more useful and easily-remembered form

$$(x - x_1)\frac{\partial S}{\partial x_1} + (y - y_1)\frac{\partial S}{\partial y_1} = 0. \quad \text{.....(2)}$$

Conjugate diameters can be very conveniently introduced by using this equation. If C is the centre of $S = 0$, and we take a series of chords parallel to PCP', the locus of their mid-points is a diameter. This diameter DCD' is the diameter conjugate to PCP'.

The chief advantage of equation (2) is that it is the equation of the chord in terms of the coordinates of Q_1 only and it does not involve the coordinates of R_1 and R_2. In spite of this, most candidates, when tackling a rider designed to illustrate the use of equation (2), almost invariably begin by saying that if R_1 is (h_1, k_1), R_2 is (h_2, k_2), then the coordinates of Q_1 are $\{\frac{1}{2}(h_1 + h_2), \frac{1}{2}(k_1 + k_2)\}$: thereby they immediately complicate all their subsequent work by introducing the irrelevant coordinates of R_1 and R_2.

Two examples will suffice to illustrate the method.

Example 1. Find the locus of the mid-points of normal chords of the parabola $y^2 = 4ax$.

The equation of the normal at $(am^2, 2am)$ is

$$y + mx = 2am + am^3. \quad \text{.....(3)}$$

The equation of the chord whose mid-point is (x_1, y_1) is

$$(y - y_1)\,2y_1 + (x - x_1)(-4a) = 0. \quad \text{.....(4)}$$

If (3) and (4) are identical,

$$\frac{y_1}{1} = \frac{-2a}{m} = \frac{y_1^2 - 2ax_1}{2am + am^3}.$$

An easy elimination of m and the dropping of the suffixes 1 gives the required locus.

If N is the point at which the normal at "m" meets the parabola again, the coordinates of N need not be found at all and do not appear in the solution.

Example II. The equation

$$x^2/a^2 - y^2/b^2 = k \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5)$$

represents a hyperbola, its asymptotes, or the conjugate hyperbola according as $k = 1, 0,$ or -1. The equation of the chord of (5) whose mid-point is (x_1, y_1) is

$$(x - x_1)\frac{x_1}{a^2} - (y - y_1)\frac{y_1}{b^2} = 0.$$

Since this is independent of k, it follows that if a chord meets a hyperbola at H_1, H_2, its asymptotes at A_1, A_2 and the conjugate hyperbola at C_1, C_2, then H_1H_2, A_1A_2, C_1C_2 all have the same mid-point. This well-known property of the hyperbola is most elegantly proved in this way.

In three-dimensional geometry the analogue of the equation of the chord whose mid-point is given is the equation of the plane section of a quadric whose centre is given. Either method applies equally in this case, and the equation of the plane section whose centre is (x_1, y_1, z_1) is

$$(x - x_1)\frac{\partial \Sigma}{\partial x_1} + (y - y_1)\frac{\partial \Sigma}{\partial y_1} + (z - z_1)\frac{\partial \Sigma}{\partial z_1} = 0,$$

where $\Sigma = 0$ is the equation of the quadric.

In particular it provides a very good method of introducing the theory of conjugate diameters of a central quadric.

Many other examples might have been given of the usefulness of this equation. If this short note has the effect of drawing the attention of some teachers to the importance and usefulness of this much-neglected equation it will have achieved its object.

E. G. P.

BUREAU FOR THE SOLUTION OF PROBLEMS.

THIS is under the direction of Mr. A. S. Gosset Tanner, M.A., 115, Radbourne Street, Derby, to whom all enquiries should be addressed, accompanied by a stamped and addressed envelope for the reply. Applicants, who must be members of the Mathematical Association, should whenever possible state the source of their problems and the names and authors of the textbooks on the subject which they possess. As a general rule the questions submitted should not be beyond the standard of University Scholarship Examinations. Whenever questions from the Cambridge Mathematical Scholarship volumes are sent, it will not be necessary to copy out the question in full, but only to send the reference, *i.e.* volume, page, and number. If, however, the questions are taken from the papers in Mathematics set to Science candidates, these should be given in full. The names of those sending the questions will not be published.

Applicants are requested to return all solutions to the Secretary.

ROTATION AND STRAIN.

By H. F. Browne.

The following method of introducing the ideas of rotation and strain into the study of Elasticity depends upon the identity

$$\mathbf{m} \cdot \mathbf{n} \text{ grad } \mathbf{u} = \mathbf{n} \cdot (\mathbf{m} \text{ grad } \mathbf{u} + \mathbf{m} \times \text{curl } \mathbf{u}) \ldots\ldots\ldots\ldots (1)$$

and upon the fact that if the straight line PQ undergoes a small rotation $\boldsymbol{\omega}$ about an axis through P, the displacement of Q is

$$\boldsymbol{\omega} \times \overrightarrow{PQ}.$$

Consider first of all a rigid body which undergoes a small displacement. Let the point P of the body move to P'. We shall represent PP' by $\mathbf{u}$ and regard $\mathbf{u}$ as typical of a vector field extending throughout the space originally occupied by the body. Let Q be a point near to P and represent $\overrightarrow{PQ}$ by $\mathbf{m}$. Then if Q moves to Q',

$$\overrightarrow{QQ'} = \mathbf{u} + \mathbf{m} \text{ grad } \mathbf{u},$$

and hence

$$\overrightarrow{P'Q'} = \mathbf{m} + \mathbf{m} \text{ grad } \mathbf{u}.$$

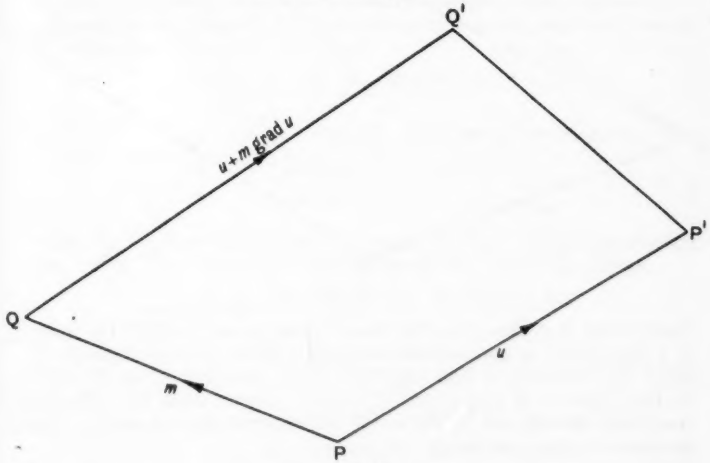


Fig. 1.

We shall suppose that $\mathbf{u}$ and $\mathbf{m}$ are of the same order of small quantities, so that $\mathbf{m}$ grad $\mathbf{u}$ is small compared with $\mathbf{m}$.

Now take any other point R near to P and let $\overrightarrow{PR} = \mathbf{n}$, so that

$$\overrightarrow{P'R'} = \mathbf{n} + \mathbf{n} \text{ grad } \mathbf{u}.$$

Then since the body is rigid and the triangles PQR, $P'Q'R'$ equal

in all respects,

$$(\mathbf{m}+\mathbf{m}\ \text{grad}\ \mathbf{u})\cdot(\mathbf{n}+\mathbf{n}\ \text{grad}\ \mathbf{u})=\mathbf{m}\cdot\mathbf{n},$$

or, neglecting $(\mathbf{m}\ \text{grad}\ \mathbf{u})\cdot(\mathbf{n}\ \text{grad}\ \mathbf{u})$,

$$\mathbf{m}\cdot\mathbf{n}\ \text{grad}\ \mathbf{u}+\mathbf{n}\cdot\mathbf{m}\ \text{grad}\ \mathbf{u}=0.$$

Hence by (1),

$$\mathbf{n}\cdot(2\mathbf{m}\ \text{grad}\ \mathbf{u}+\mathbf{m}\times\text{curl}\ \mathbf{u})=0,$$

or since $\mathbf{n}$ is arbitrary,

$$\mathbf{m}\ \text{grad}\ \mathbf{u}=(\tfrac{1}{2}\ \text{curl}\ \mathbf{u})\times\mathbf{m}.$$

We now have

$$\overrightarrow{P'Q'}=\mathbf{m}+(\tfrac{1}{2}\ \text{curl}\ \mathbf{u})\times\mathbf{m},$$

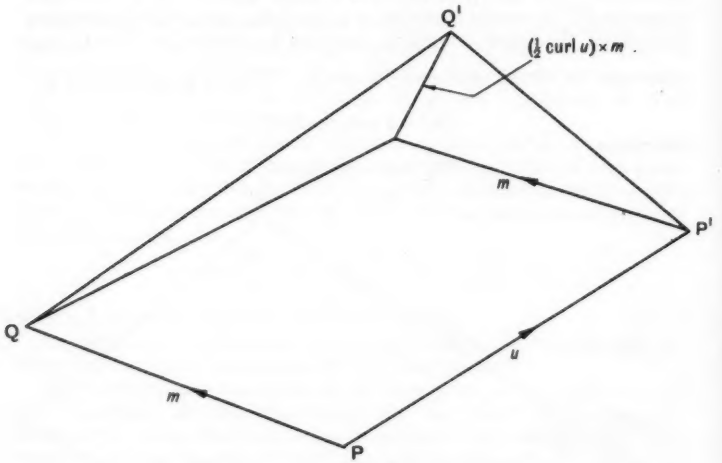


FIG. 2.

from which it is clear that the line PQ has moved to $P'Q'$ by means of a translation $\mathbf{u}$ and a rotation $\frac{1}{2}$ curl $\mathbf{u}$ about an axis through P'. Since the rotation is half the curl of the vector field at P it will be the same for all short lines PQ, PR, ... drawn from P. (The fact that the rotation can be shown to be uniform throughout the field need not concern us here.)

Next consider the case of an elastic body which undergoes a small displacement. As before, let a point P of the body move to P' and let $PP'=\mathbf{u}$. The part of the body in the neighbourhood of P moves into the neighbourhood of P', but it becomes distorted whilst doing so. Let us suppose that the movement of this part of the body takes place in two stages, first a rigid body movement consisting of a translation $\mathbf{u}$ and a rotation $\frac{1}{2}$ curl $\mathbf{u}$, and secondly a distortion.

In the first stage $\overrightarrow{PQ}$ changes from $\mathbf{m}$ to $\mathbf{m}+(\frac{1}{2}\ \text{curl}\ \mathbf{u})\times\mathbf{m}$, and in

the second stage it changes to $\mathbf{m} + \mathbf{m}\,\text{grad}\,\mathbf{u}$. Hence the displacement of Q relative to P in the second stage is

$$\mathbf{m}\,\text{grad}\,\mathbf{u} - (\tfrac{1}{2}\,\text{curl}\,\mathbf{u}) \times \mathbf{m}.$$

We shall call this the " strain of Q relative to P ".

We see that the " rotation at P " is the vector $\frac{1}{2}\,\text{curl}\,\mathbf{u}$, and that the " strain of Q relative to P " is obtained by substituting $\overrightarrow{PQ}$ for $\boldsymbol{\xi}$ in the linear vector function

$$e(\boldsymbol{\xi}) = \boldsymbol{\xi}\,\text{grad}\,\mathbf{u} + \tfrac{1}{2}\boldsymbol{\xi} \times \text{curl}\,\mathbf{u}.$$

The fact that this function is symmetrical, *i.e.* that

$$\boldsymbol{\eta} \cdot e(\boldsymbol{\xi}) = \boldsymbol{\xi} \cdot e(\boldsymbol{\eta})$$

follows at once by substituting $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ for $\mathbf{m}$ and $\mathbf{n}$ in (1).

These results may be expressed in cartesian coordinates by means of the usual formulae. Thus if $\mathbf{u} = \iota_1 u_1 + \iota_2 u_2 + \iota_3 u_3$, then the strain function

$$e(\boldsymbol{\xi}) = \Sigma e_{jk} \iota_j \xi_k$$

where

$$e_{jk} = \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k}.$$

Evidently $e_{jk} = e_{kj}$, and the symmetry of $e(\boldsymbol{\xi})$ is again apparent.

To relate this idea of strain to the more familiar physical notions we define the " stretch " of PQ as

$$\begin{aligned} 1 - \frac{P'Q'}{PQ} &= 1 - \frac{\mathbf{m} \cdot (\mathbf{m} + \mathbf{m}\,\text{grad}\,\mathbf{u})}{m^2} \\ &= \mathbf{m} \cdot e(\mathbf{m})/m^2 \quad \text{since } \mathbf{m} \cdot \mathbf{m} \times \text{curl}\,\mathbf{u} = 0, \\ &= e_{jk} m_j m_k / m^2 \\ &= e_{jk} m'_j m'_k, \end{aligned}$$

where the m''s are the direction-cosines of PQ. Further, we define the " slide " or " shear " of the perpendicular lines PQ, PR as

$$\begin{aligned} \cos Q'P'R' &= \frac{(\mathbf{m} + \mathbf{m}\,\text{grad}\,\mathbf{u}) \cdot (\mathbf{n} + \mathbf{n}\,\text{grad}\,\mathbf{u})}{mn} \\ &= \frac{\mathbf{m} \cdot \mathbf{n}\,\text{grad}\,\mathbf{u} + \mathbf{n} \cdot \mathbf{m}\,\text{grad}\,\mathbf{u}}{mn}, \\ &\qquad \text{neglecting } (\mathbf{m}\,\text{grad}\,\mathbf{u}) \cdot (\mathbf{n}\,\text{grad}\,\mathbf{u}) \\ &= \mathbf{n} \cdot (2\mathbf{m}\,\text{grad}\,\mathbf{u} + \mathbf{m} \times \text{curl}\,\mathbf{u})/mn \qquad \text{by (1)} \\ &= 2\mathbf{n} \cdot e(\mathbf{m})/mn \\ &= 2 e_{jk} m'_j n'_k. \end{aligned}$$

By taking PQ and PR in the directions of ι_1, ι_2, ι_3, we find that the six tensor elements e_{jk} are composed of the three stretches in the directions of the coordinate axes, and the halves of the three slides parallel to the axes.

H. F. B.

CARTESIAN COORDINATES IN TRIANGLE GEOMETRY.

By R. Sibson.

The points A, B, C on the circle $x^2+y^2=R^2$ are

$$(R\cos\theta_i,\ R\sin\theta_i) \quad (i=1, 2, 3).$$

We may without loss of generality assume $0 \leqslant \theta_1 < \theta_2 < \theta_3 < 2\pi$.

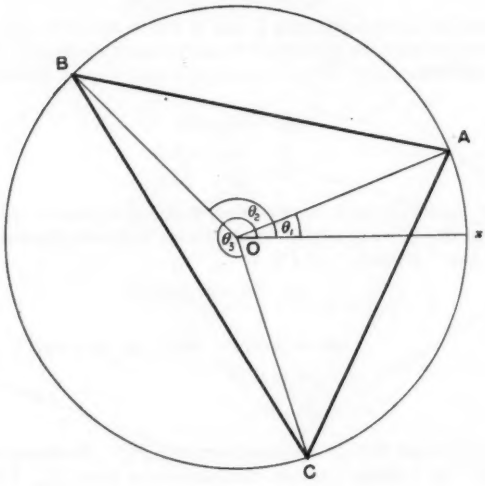


Fig. 1.

In the figure,
$$\theta_3-\theta_2=2\angle A,$$
$$2\pi-(\theta_3-\theta_1)=2\angle B,$$
$$\theta_2-\theta_1=2\angle C;$$

or
$$\angle A=\tfrac{1}{2}(\theta_3-\theta_2),\ \angle B=\pi-\tfrac{1}{2}(\theta_3-\theta_1),\ \angle C=\tfrac{1}{2}(\theta_2-\theta_1).$$

The lack of symmetry in these results appears inevitable, but is of no consequence in the majority of results. It becomes important in results in which the in-centre and ex-centres and the related radii are concerned, where the above results are necessary to distinguish between the points I, I_1, I_2, I_3.

The reader may be interested to see for himself that the above results are still true when O falls outside the triangle ABC.

The centroid G of the triangle ABC has coordinates

$$G \equiv \{\tfrac{1}{3}R(\cos\theta_1+\cos\theta_2+\cos\theta_3),\ \tfrac{1}{3}R(\sin\theta_1+\sin\theta_2+\sin\theta_3)\}. \quad \ldots\text{I}$$

Now the chord AB has the equation

$$x\cos\tfrac{1}{2}(\theta_1+\theta_2)+y\sin\tfrac{1}{2}(\theta_1+\theta_2)=R\cos\tfrac{1}{2}(\theta_1-\theta_2).$$

Hence the altitude CF through C has the equation

$$(y-R\sin\theta_3)\cos\tfrac{1}{2}(\theta_1+\theta_2)=(x-R\cos\theta_3)\sin\tfrac{1}{2}(\theta_1+\theta_2),$$

or $$y\cos\tfrac{1}{2}(\theta_1+\theta_2)-x\sin\tfrac{1}{2}(\theta_1+\theta_2)=R\sin[\theta_3-\tfrac{1}{2}(\theta_1+\theta_2)]. \quad \ldots\ldots\text{(i)}$$

Similarly the altitude BE has the equation

$$y\cos\tfrac{1}{2}(\theta_1+\theta_3)-x\sin\tfrac{1}{2}(\theta_1+\theta_3)=R\sin[\theta_2-\tfrac{1}{2}(\theta_3+\theta_1)]. \quad \ldots\ldots\text{(ii)}$$

Solving equations (i) and (ii) we get

$$\begin{aligned} y\{&\cos\tfrac{1}{2}(\theta_1+\theta_2)\sin\tfrac{1}{2}(\theta_1+\theta_3)-\cos\tfrac{1}{2}(\theta_1+\theta_3)\sin\tfrac{1}{2}(\theta_1+\theta_2)\} \\ &=R\{\sin[\theta_3-\tfrac{1}{2}(\theta_1+\theta_2)]\sin\tfrac{1}{2}(\theta_1+\theta_3) \\ &\quad -\sin[\theta_2-\tfrac{1}{2}(\theta_3+\theta_1)]\sin\tfrac{1}{2}(\theta_1+\theta_2)\}, \end{aligned}$$

or $$\begin{aligned} y\sin\tfrac{1}{2}(\theta_3-\theta_2)&=\tfrac{1}{2}R\{\cos[\tfrac{1}{2}(\theta_3-\theta_2)-\theta_1]-\cos\tfrac{1}{2}(3\theta_3-\theta_2) \\ &\quad -\cos[\tfrac{1}{2}(\theta_2-\theta_3)-\theta_1]+\cos\tfrac{1}{2}(3\theta_2-\theta_3)\} \\ &=R\{-\sin\theta_1\sin\tfrac{1}{2}(\theta_2-\theta_3)+\sin\tfrac{1}{2}(\theta_2+\theta_3)\sin(\theta_3-\theta_2)\} \\ &=R\{\sin\theta_1\sin\tfrac{1}{2}(\theta_3-\theta_2)+2\sin\tfrac{1}{2}(\theta_2+\theta_3)\sin\tfrac{1}{2}(\theta_3-\theta_2)\cos\tfrac{1}{2}(\theta_3-\theta_2)\}. \end{aligned}$$

Thus $$\begin{aligned} y&=R\{\sin\theta_1+2\sin\tfrac{1}{2}(\theta_2+\theta_3)\cos\tfrac{1}{2}(\theta_2-\theta_3)\} \\ &=R(\sin\theta_1+\sin\theta_2+\sin\theta_3). \end{aligned}$$

Similarly $$x=R(\cos\theta_1+\cos\theta_2+\cos\theta_3).$$

Thus the coordinates of the orthocentre H are

$$H \equiv \{R(\cos\theta_1+\cos\theta_2+\cos\theta_3),\quad R(\sin\theta_1+\sin\theta_2+\sin\theta_3)\}. \quad \ldots\text{II}$$

Results I and II show that O, G and H are collinear and $OG=\frac{1}{3}OH$.

The coordinates of the mid-points A', B', C' of the sides BC, CA, AB of the triangle are given by

$$A' \equiv \{\tfrac{1}{2}R(\cos\theta_2+\cos\theta_3),\quad \tfrac{1}{2}R(\sin\theta_2+\sin\theta_3)\},\ \text{etc.}$$

Now consider the point N', where

$$N' \equiv \{\tfrac{1}{2}R(\cos\theta_1+\cos\theta_2+\cos\theta_3),\ \tfrac{1}{2}R(\sin\theta_1+\sin\theta_2+\sin\theta_3)\}.$$

Then $$A'N'^2=\tfrac{1}{4}R^2(\cos^2\theta_1+\sin^2\theta_1)=\tfrac{1}{4}R^2,$$

and hence $A'N'=\frac{1}{2}R=B'N'=C'N'$, similarly.

Thus N' is the circumcentre of the triangle $A'B'C'$.

Now D, the foot of the altitude AD, has coordinates which satisfy the equations (*cf.* (i))

$$x\cos\tfrac{1}{2}(\theta_2+\theta_3)+y\sin\tfrac{1}{2}(\theta_2+\theta_3)=R\cos\tfrac{1}{2}(\theta_2-\theta_3),$$

$$x\sin\tfrac{1}{2}(\theta_2+\theta_3)-y\cos\tfrac{1}{2}(\theta_2+\theta_3)=R\sin[\tfrac{1}{2}(\theta_2+\theta_3)-\theta_1].$$

Solving for x and y, we have

$$\begin{aligned} x&=R\{\cos\tfrac{1}{2}(\theta_2-\theta_3)\cos\tfrac{1}{2}(\theta_2+\theta_3)+\sin[\tfrac{1}{2}(\theta_2+\theta_3-\theta_1]\sin\tfrac{1}{2}(\theta_2+\theta_3)\} \\ &=\tfrac{1}{2}R\{\cos\theta_2+\cos\theta_3+\cos\theta_1-\cos(\theta_2+\theta_3-\theta_1)\}, \end{aligned}$$

and similarly for y. *Thus the coordinates of D, the foot of the perpendicular from A to BC, are*

$$D \equiv \{\tfrac{1}{2}R(\cos\theta_1+\cos\theta_2+\cos\theta_3-\cos(\theta_2+\theta_3-\theta_1)) \quad \tfrac{1}{2}R(\sin\theta_1+\sin\theta_2+\sin\theta_3-\sin(\theta_2+\theta_3-\theta_1))\}. \quad \text{.....III}$$

(A clumsy-looking result, but wait!).

Hence $DN'^2=\tfrac{1}{4}R^2\{\cos^2(\theta_2+\theta_3-\theta_1)+\sin^2(\theta_2+\theta_3-\theta_1)\}=\tfrac{1}{4}R^2$,

and $DN'=\tfrac{1}{2}R$.

Thus the circle $A'B'C'$ passes through D, E, F.

The mid-point α of AH has coordinates

$$\alpha \equiv \{\tfrac{1}{2}R(2\cos\theta_1+\cos\theta_2+\cos\theta_3),\ \tfrac{1}{2}R(2\sin\theta_1+\sin\theta_2+\sin\theta_3)\}.$$

Hence $N'\alpha^2=\tfrac{1}{4}R^2(\cos^2\theta_1+\sin^2\theta_1)=\tfrac{1}{4}R^2$.

Thus $N'\alpha=\tfrac{1}{2}R=N'\beta=N'\gamma$, where β and γ are similarly defined. *Hence the same circle passes through A', B', C', D, E, F, α, β, γ and its radius is $\tfrac{1}{2}R$ and its centre is N' where*

$$N' \equiv \{\tfrac{1}{2}R(\cos\theta_1+\cos\theta_2+\cos\theta_3),\ \tfrac{1}{2}R(\sin\theta_1+\sin\theta_2+\sin\theta_3)\}. \quad \text{...IV}$$

Results II and IV show that O, N', and H are collinear and that $ON'=\tfrac{1}{2}OH$.

Now consider the point X, (Fig. 2), where

$$X \equiv \{-R\cos(\theta_2+\theta_3-\theta_1),\ -R\sin(\theta_2+\theta_3-\theta_1)\}.$$

The mid-point of HX is D, by results II and III. But X is on the circle $x^2+y^2=R^2$.

Thus the circle is cut again by AD at X, and $HD=DX$, where

$$X \equiv \{-R\cos(\theta_2+\theta_3-\theta_1),\ -R\sin(\theta_2+\theta_3-\theta_1)\}. \quad \text{.....V}$$

Now let P be a fourth point on the circle ABC, where

$$P \equiv (R\cos\theta_4,\ R\sin\theta_4).$$

The mid-point T of PH is given by

$$T \equiv \{\tfrac{1}{2}R\sum_1^4\cos\theta_i,\ \tfrac{1}{2}R\sum_1^4\sin\theta_i\}.$$

Now these are symmetrical in θ_1, θ_2, θ_3, θ_4, so that T is also the mid-point of the join of A to the orthocentre of the triangle PBC, and so on.

Hence the joins of four concyclic points to the orthocentres of the triangles formed by the remaining three points are concurrent and bisect one another.

Also $TN'^2=\tfrac{1}{4}R^2(\cos^2\theta_4+\sin^2\theta_4)=\tfrac{1}{4}R^2$,

so that $TN'=\tfrac{1}{2}R$.

Hence T lies on the nine-points circle of ABC and similarly on the nine-points circles of PBC, PCA, PAB.

Thus the nine-points circles of the four triangles formed from four concyclic points are concurrent.

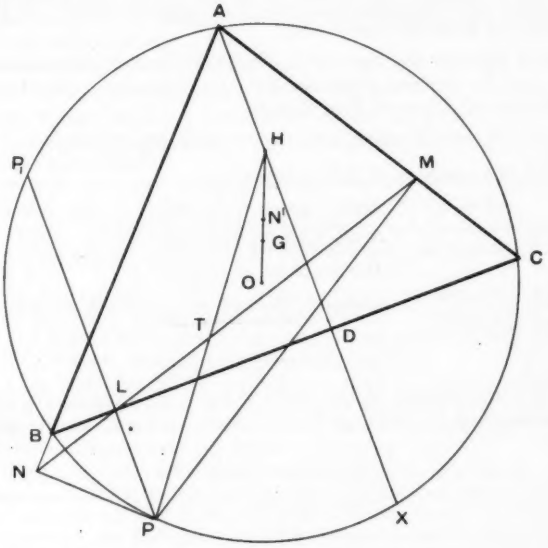


FIG. 2.

Now if PL, PM, PN are the perpendiculars from P to BC, CA, AB, then we can write down the coordinates of L immediately by replacing θ_1 by θ_4 in the coordinates of D (result III): that is,

$$L \equiv \{\tfrac{1}{2}R(\cos\theta_2+\cos\theta_3+\cos\theta_4-\cos(\theta_2+\theta_3-\theta_4),$$
$$\tfrac{1}{2}R(\sin\theta_2+\sin\theta_3+\sin\theta_4-\sin(\theta_2+\theta_3-\theta_4))\}.$$

Hence the gradient of LT is equal to

$$\frac{\sin\theta_1+\sin(\theta_2+\theta_3-\theta_4)}{\cos\theta_1+\cos(\theta_2+\theta_3-\theta_4)}$$
$$=\frac{2\sin\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_4)\cos\tfrac{1}{2}(\theta_1+\theta_4-\theta_2-\theta_3)}{2\cos\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_4)\cos\tfrac{1}{2}(\theta_1+\theta_4-\theta_2-\theta_3)}$$
$$=\tan\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_4).$$

Now this is symmetrical in θ_1, θ_2, θ_3, so that LT, MT, NT have the same gradient. Thus L, M, N, T are collinear.*

* *Cf.* Niewenglowski, *Cours de Géométrie Analytique*, 1911 (2), I, p. 196, ex. 5.

Hence the Simson line property is proved, and also the fact that the Simson line LMN of P with respect to the triangle ABC passes through T, that is, bisects the line joining P to the orthocentre of the triangle ABC.

Similarly the Simson lines of each of the other points with respect to the triangle formed by the remaining three pass through T, so that the four Simson lines are concurrent.

Now if PL cuts the circle $x^2+y^2=R^2$ again at P_1, the coordinates of P_1 can be written down at once by replacing θ_1 by θ_4 in the coordinates of X (result V); that is,

$$P_1 \equiv \{-R\cos(\theta_2+\theta_3-\theta_4),\ -R\sin(\theta_2+\theta_3-\theta_4)\}.$$

Hence the gradient of AP_1 is equal to

$$\frac{\sin\theta_1+\sin(\theta_2+\theta_3-\theta_4)}{\cos\theta_1+\cos(\theta_2+\theta_3-\theta_4)}$$

$$=\text{the gradient of } LMN.$$

*Hence if P_2, P_3 are similarly defined, the lines LMN, AP_1, BP_2, CP_3 are parallel.**

Also if $$Q \equiv (R\cos\theta_5,\ R\sin\theta_5)$$

and PQ is a diameter of the circle $x^2+y^2=R^2$, then $\theta_5=\theta_4\pm\pi$ and the gradient of the Simson line of Q with respect to the triangle ABC is

$$\begin{aligned}&\tan\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_5)\\ &=\tan\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_4\pm\pi)\\ &=-\cot\tfrac{1}{2}(\theta_1+\theta_2+\theta_3-\theta_4).\end{aligned}$$

Hence the product of the gradients of the Simson lines of P and Q is equal to -1.

Thus the Simson lines of P and Q are at right angles.

Also $$\begin{aligned}P_2P_3^2 &= R^2[\{\cos(\theta_3+\theta_1-\theta_4)-\cos(\theta_1+\theta_2-\theta_4)\}^2\\ &\qquad+\{\sin(\theta_3+\theta_1-\theta_4)-\sin(\theta_1+\theta_2-\theta_4)\}^2]\\ &=R^2\{2-2\cos(\theta_3-\theta_2)\},\end{aligned}$$

and $$\begin{aligned}BC^2 &= R^2\{(\cos\theta_2-\cos\theta_3)^2+(\sin\theta_2-\sin\theta_3)^2\}\\ &=R^2\{2-2\cos(\theta_3-\theta_2)\}.\end{aligned}$$

Hence $BC=P_2P_3$, and similarly $CA=P_3P_1$, $AB=P_1P_2$.

Thus the triangles $P_1P_2P_3$, ABC are congruent.

Now if we remove the restriction that Q is at the opposite end of the diameter through P, and if we call T now T_5 and define T_1, T_2, T_3, T_4 similarly, we can proceed to the following result.

Consider $$U \equiv \{\tfrac{1}{2}R\sum_1^5\cos\theta_i,\ \ \tfrac{1}{2}R\sum_1^5\sin\theta_i\}.$$

* *Cambridge Scholarship Papers*, cxxxiv, p. 22, no. 1.

$$UT_1^2 = \tfrac{1}{4}R^2\{\cos^2\theta_1 + \sin^2\theta_1\} = \tfrac{1}{4}R^2.$$

Thus $$UT_1 = \tfrac{1}{2}R = UT_2 = UT_3 = UT_4 = UT_5.$$

Hence the five T-points are on a circle centre U, radius $\frac{1}{2}R$.

In other words, *the nine-points circles of the ten triangles formed by five concyclic points, being ten equal circles, are concurrent in fours at five points which are concyclic on a circle of the same radius as each of the nine-points circles.*

If we introduce a sixth point θ_6, *the six U-circles obtained from the six possible sets of five points are concurrent, and so on.*

The figure showing the concurrencies of the ten nine-points circles can be completed to include sixteen equal circles meeting in fives at sixteen different points! R. S.

1365. I came here (*i.e.* to Blundell's School), taught by my mother . . . with no other supply of knowledge than an acquaintance with the Latin Grammar . . . and a slight knowledge of the books of Euclid and of Bonnycastle's Arithmetic. (p. 31.)

1366. On September 8, 1835, he writes: "I do not think Mr. Mills (a teacher at Blundell's) a very good mathematician; I proposed doing the eleventh and twelfth books after I had done the rest, but to my astonishment I found that he knew very little about them." (p. 31.)

1367. I can remember that I got hold of a book . . . Bland's Algebraical Problems, and worked entirely through it in my play-time, and found afterwards that it gave me a mastery of the lower parts of algebra which I doubt if I could have obtained from any teacher whatsoever. (p. 35.)

1368. Temple, by his own studies at school, came up considerably ahead of his contemporaries in mathematics, and W. G. Ward most kindly took him alone, to look over his work once a week for an hour. Ward was brilliant in pure mathematics, but confessedly unable to help his pupils in applied mathematics. In 1841 Temple writes: "The only thing I can complain of in Oxford is the low state of science. They have not a single book on logic worth reading, and metaphysics and moral philosophy are at a very low ebb. Mathematics are still lower." Mr. Ward . . . looked over his exercises, correcting them with extraordinary rapidity, showing here and there easier methods, noting interesting mathematical points in the various possible mathematical solutions. But he goes on to say that the hour was about equally divided between algebra or the differential calculus and conversation, sometimes on the drama or opera, but later on, as Ward attached himself to the Tractarian party, more often on theology. . . . (p. 51.)

1369. The curriculum of study for a first-class in mathematics (at Oxford) prior to 1852 embraced only about two-thirds of the amount then required at Cambridge. The mathematical papers set at that time included, besides all the elementary subjects and the differential and integral calculus, some sections of Newton with problems deducible, and mechanics, including rigid dynamics, with hydrostatics, geometrical and physical optics, and astronomy. Cambridge mathematicians who migrated to Oxford Fellowships used to regard an Oxford First Class in Mathematics at that time to be only equivalent to a high Senior Optime or a low Wrangler. The problems set demanded, however, a thorough grasp of principles, but no great analytical dexterity. (p. 38)

Gleanings 1365-1369 from *Memoirs of Archbishop Frederick Temple, I.* [Per Mr. A. F. Mackenzie.]

ON SUMS OF DIGITS.

By Eric Goodstein.

Introduction.

If we form the sum $S(n)$ of the digits of a number n, and then the sum of the digits of $S(n)$, and so on, in a finite number of steps we reach a number of one digit, $R(n)$, which we call the reduce of n.

Starting with a number prime to 9, say 2, consider the reduces of powers of 2; we obtain the sequence 2, 4, 8, 7, 5, 1, 2, 4, 8, ..., which recurs after the sixth term. Consider next a five-digit number, say 16427; from this we derive 15794 formed by replacing each digit of 16427 by the reduce of the sum of the remaining digits. Repeating the transformation, we obtain in turn 73184, 72461, 49751, 48137, 16427, the number being restored to its original value after only six transformations. It will be found that a three-digit number repeats after eighteen transformations and yet for a twenty-digit number only six transformations are needed.

That there must be some recurrences in a sufficiently long series of transformations is obvious, since (in the scale of 10) there are less than 10^n numbers of n digits, but it is by no means obvious that the initial number must recur, and in fact this is not always true, since a general number of four digits will not recur. What is perhaps more surprising is that the number of transformations required is so small.

We seek for the explanation of these recurrences and prove some theorems on the recurrence of numbers with any assigned number of digits in an assigned scale of notation.

Definitions.

Any integer x can be expressed uniquely in the form

$$a_p(n+1)^p + a_{p-1}(n+1)^{p-1} + \ldots + a_2(n+1)^2 + a_1(n+1) + a_0,$$

where $a_k < n+1$, $0 \leqslant k \leqslant p$.

We make the definitions

$$S_n^0(x) = a_0 + a_1 + \ldots + a_{p-1} + a_p,$$

$$S_n^{r+1}(x) = S_n^0(S_n^r(x)),$$

and $R_n(x) = S_n^t(x)$ for the least t for which $S_n^t(x) < n+1$.

It follows from the definition that

$$R_n(R_n(x)) = R_n(x).$$

Since $x - S_n^0(x) = a_p\{(n+1)^p - 1\} + a_{p-1}\{(n+1)^{p-1} - 1\} + \ldots + a_1\{(n+1) - 1\}$

we have

$$S_n^0(x) \equiv x \pmod{n},$$

$$S_n^1(x) \equiv S_n^0(x) \pmod{n},$$

$$S_n^2(x) \equiv S_n^1(x) \pmod{n},$$
$$\vdots$$
$$R_n(x) = S_n^t(x) \equiv S_n^{t-1}(x) \pmod{n}.$$

Thus $R_n(x) \equiv x \pmod{n}$,

that is, $x - R_n(x) = M(n)$. .....(i)

Thus if k is the greatest integer such that $kn < x$, then

$$R_n(x) = x - kn.$$

From (1), by taking $x + M(n)$ for x, we obtain

$$x + M(n) - R_n(x + M(n)) = M(n),$$

and subtracting (i) from this,

$$R_n(x + M(n)) - R_n(x) = M(n).$$

But $|R_n(x + M(n)) - R_n(x)| < n$,

and so $R_n(x + M(n)) - R_n(x) = 0$,

or $R_n(x + M(n)) = R_n(x)$.

Reductions of sums and products.

Since $R_n(x) \equiv x \pmod{n}$

and $R_n(y) \equiv y \pmod{n}$,

$$kx + ly \equiv kR_n(x) + lR_n(y) \pmod{n},$$

and so $R_n(kx + ly) = R_n\{kR_n(x) + lR_n(y)\}$.

Also $xy \equiv R_n(x) \,.\, R_n(y) \pmod{n}$,

and therefore $R_n(xy) = R_n\{R_n(x) \,.\, R_n(y)\}$. .....(ii)

If $\phi(n)$ is the number of integers less than and prime to n (including unity), and if a is prime to n,

$$R_n\{a^{\phi(n)}\} = 1.$$

This is an immediate consequence of Euler's extension of Fermat's theorem, namely

$$a^{\phi(n)} \equiv 1 \pmod{n}.$$

Thus
$$\begin{aligned} R_n\{a^{p+m(\phi(n))}\} &= R_n\{R_n(a^p)\} \,.\, R_n\{a^{m(\phi(n))}\} \\ &= R_n\{R_n(a^p)\} \\ &= R_n(a^p). \end{aligned}$$

But by repeated application of (ii),

$$R_n(a^b) = R_n\{[R_n(a)]^b\}.$$

Therefore $R_n\{a^{p+m(\phi(n))}\} = R_n\{[R_n(a)]^p\}$,

or $R_n(a^b) = R_n\{[R_n(a)]^\rho\}$

where $\rho = R_{\phi(n)}(b)$.

If $q_1, q_2, \ldots q_n$ are the prime factors of n, then

$$\phi(N) = N\left(1-\frac{1}{q_1}\right)\left(1-\frac{1}{q_2}\right)\ldots\left(1-\frac{1}{q_n}\right).$$

If $N>2$, at least one q is odd and for this q, the number $q-1$ is even and so $\phi(N)$ is even.

Let $R_n(a^s) = a_s$, where a is prime to n.

$$\begin{aligned} a_r a_{\phi(n)-r} &= R_n(a^r) \,.\, R_n(a^{\phi(n)-r}) \\ &\equiv a^{\phi(n)} \pmod n \\ &\equiv 1 \pmod n. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iii)} \end{aligned}$$

Let $r = \frac{1}{2}\phi(n)$, where r is an integer since $\phi(n)$ is even if $n>2$.

Then by (iii) $\quad \{a_{\phi(n)/2}\}^2 \equiv 1 \pmod n$,

whence $\quad a_{\phi(n)/2} \equiv \pm 1 \pmod n$,

and hence $\quad a_{\phi(n)/2} = 1$ or $n-1$.

If $\quad a_{\phi(n)/2} = 1$,

$$\begin{aligned} a_{\{\phi(n)/2\}+r} &= R_n\{R_n(a^{\phi(n)/2}) \,.\, R_n(a^r)\} \\ &= R_n(a^r) \\ &= a_r. \end{aligned}$$

If $\quad a_{\phi(n)/2} = n-1$,

$$\begin{aligned} a_{\{\phi(n)/2\}+r} + a_r &= R_n(a^{\{\phi(n)/2\}+r}) + R_n(a^r) \\ &\equiv a^r(a^{\phi(n)/2}+1) \pmod n \\ &= na^r \pmod n \\ &= M(n). \end{aligned}$$

But $0<a_k<n$, strict inequality since a is prime to n, and so if

$$\begin{aligned} a_{\phi(n)/2} &= n-1 \\ a_{\{\phi(n)/2\}+r} + a^r &= n. \end{aligned}$$

Theorem. If $x = a_p^0(n+1)^p + a_{p-1}^0(n+1)^{p-1} + \ldots + a_1^0(n+1) + a_0^0$,

we define $\quad a_k^{r+1} = R_n(S_r - a_k^r)$,

where $\quad S_r = a_0^r + a_1^r + \ldots + a_p^r$,

and $\quad T_n^r(x) = a_p^r(n+1)^p + a_{p-1}^r(n+1)^{p-1} + \ldots + a_1^r(n+1) + a_0^r$.

Then, provided $a_r^0 > 0$, $0 \leqslant r \leqslant p$, we prove

1. If the highest common factor of S_0 and $n(p+1)$ is unity, then,
   - (α) if n is prime to $p(p+1)$, $T_n^{\phi(n)}(x) = x$;
   - (β) if n is prime to p but n and $p+1$ have a common factor, $T_n^{\phi\{n(p+1)\}}(x) = x$;
   - (γ) if n and p have a common factor, then for all $r>0$, $T_n^r(x) \neq x$.

2. If the highest common factor of S_0 and $n(p+1)$ is $c>1$ and $n(p+1)=cN$, then,

(α) if N is prime to p, $T_n^{\phi(N)}(x)=x$;

(β) if N is not prime to p, then for all $r>0$, $T_n^r(x)\neq x$.

The method of construction of the numbers a_k^r shows that

$$a_k^0=a_k^0;$$

$$a_k^1=R_n\left(\frac{p+1}{p+1}\cdot S_0-a_k^0\right);$$

$$a_k^2=R_n\left(\frac{p^2-1}{p+1}\cdot S_0+a_k^0\right);$$

$$a_k^3=R_n\left(\frac{p^3+1}{p+1}\cdot S_0-a_k^0\right);$$

$$\vdots$$

$$a_k^{2r}=R_n\left(\frac{p^{2r}-1}{p+1}\cdot S_0+a_k^0\right).$$

Now if we require $a_k^{2r}=a_k^0$, we must have

$$\frac{p^{2r}-1}{p+1}\cdot S_0\equiv 0 \pmod{n}.$$

1. If S_0 and $n(p+1)$ are co-prime we require

$$\frac{p^{2r}-1}{p+1}\equiv 0 \pmod{n}:$$

(α) if n is prime to $p(p+1)$, the congruence is satisfied if $p^{2r}-1\equiv 0 \pmod{n}$,

and therefore, by Euler's theorem, if $2r=\phi(n)$.

If $n=1$ or 2, $\phi(n)$ is odd and we must take $2r=2\phi(n)$.

(β) if n and $p+1$ have a common factor, but n is prime to p, then $n(p+1)$ is prime to p and so the congruence

$$p^{2r}-1=0 \pmod{n(p+1)}$$

is satisfied by $2r=\phi(n(p+1))$.

(γ) if n and p have a common factor c, then $p^{2r}-1$ is not divisible by c and so not divisible by n and the congruence has no solution.

2. If S_0 and $n(p+1)$ have a common factor $c>1$, and $n(p+1)=cN$ then we require

$$(p^{2r}-1)S_0=0 \pmod{n(p+1)}$$

that is, $$p^{2r}-1=0 \pmod{N}.$$

which is satisfied, if N is prime to p, by $2r=\phi(N)$.

If N and p have a common factor we have shown that the congruence has no solution. E. G.

MATHEMATICAL NOTES.

1520. *The Reverend Thomas Bayes, F.R.S.*

Thomas Bayes was a son of, and shares the same grave in Bunhill fields with, the Reverend Joshua Bayes (1671–1746), an eminent nonconformist divine about whom there is an article in the *Dictionary of National Biography* (which does not mention Thomas). Had this relationship been known it would have been less difficult to irradiate the obscurity that has since enshrouded the son, who seems to have courted it for he wrote anonymously.

Nor does his name appear among the alumni of the colleges of Oxford and Cambridge; but Musgrave's "Obituary", describing him as a "presbyterian theologian", states that he died at Tunbridge Wells. There seems to be a discrepancy when, a century later, we find Professor Augustus De Morgan including Bayes as one of the three great scientific unitarian ministers of the eighteenth century, the others being Priestley the chemist and Price the actuary. We have come to associate presbyterianism with calvinistic trinitarianism but the early English presbyterians—who had little, if any, connection with the present Presbyterian Church of England—were so called because of their form of church-government by presbyteries, which form they afterwards deserted towards that of the independents or congregationalists, i.e. towards the autonomy of each congregation. As is stated in the fourteenth edition of the *Encyclopaedia Britannica*, "From the beginning of the 18th century the greater part of the presbyterian congregations became practically independent in polity and unitarian in doctrine." The fact that he was educated at neither Oxford nor Cambridge and the sponsorship by the Reverend Dr. Richard Price of Bayes's paper, read posthumously before the Royal Society, might both have been regarded as presumptions that he was a dissenter.

Musgrave's references are to (*a*) the *London Magazine*, which records the death in April 1761 of the "Rev. Mr. Thomas Bayes, a dissenting minister", and (*b*) the *Gentleman's Magazine*, which records similarly the death of the "Rev. Mr. Bayes at Tunbridge Wells". The epitaph in Bunhill fields gives his age at death as 59, so that he must have been born in 1701 or 1702.

He was a native of London and got a liberal education for the ministry: and his first "settlement" was in the chapel in Leather lane as assistant to his father. Afterwards he became pastor for many years of a dissenting congregation in Tunbridge Wells. William Whiston met him there in 1746 and, in the second edition of his memoirs (1753, while Bayes was still alive), wrote of him as "a man of considerable learning and judgment, of excellent natural abilities and a good mathematician".

The British Museum possesses (1) an anonymous brochure by Thomas Bayes on "Divine Benevolence" published in 1731 by John Noon which, we are told in Walter Wilson's *Dissenting*

Churches, "attracted notice and was held in high esteem", and (2) a recent German work dealing with Bayes. Dr. Williams's Library, which is slightly older than that of the British Museum, possesses (1) "Divine Benevolence" and (2) another anonymous brochure of 52 pages, published in 1736 also by Noon, entitled "An introduction to the doctrine of Fluxions and defence of the Mathematicians against the Objections of the Author of the Analyst so far as they are designed to affect their general Methods of Reasoning."

In *Notes and Queries* of January 7, 1860, there was an informative query, desiring further information, about Thomas Bayes. It was written by Augustus De Morgan, a great-grandson of James Dodson, the founder of scientific life assurance. In the opinion of De Morgan: (*a*) the fame of Bayes will rest chiefly upon his paper in the *Philosophical Transactions* for 1763 and the supplement in the volume for 1764, which are the mathematical foundation of that branch of the theory of probabilities in which the probabilities are a matter of calculation from past events; (*b*) Laplace did not make adequate acknowledgment of his indebtedness to Bayes but more justice was done to him by Dr. C. Gourand in 1848 in his *Histoire du Calcul des Probabilités*; (*c*) Bayes was of the calibre of De Moivre—an estimate not shared by Isaac Todhunter—and Laplace in his power over the subject of probabilities; and (*d*) Dr. Price appears as a far more powerful mathematician in his explanations and comments upon Bayes than in any other part of his writings.

If this was the verdict upon Thomas Bayes during the Victorian era's *post-hoc-ergo-propter-hoc* attitude towards the physical universe, he ought to be far more appreciated to-day when we are less interested in theories of cause and effect, when our pre-occupation is rather with the facts of frequencies, when we have been taught by Professor Dirac of Cambridge that:

"When an observation is made on any atomic system that has been prepared in a given way and is thus in a given state, the result will not in general be determinate, i.e. if the experiment is repeated several times under identical conditions several different results may be obtained. If the experiment is repeated a large number of times it will be found that each particular result will be obtained a definite fraction of the total number of times, so that one can say there is a definite probability of its being obtained any time the experiment is performed."

J. G. ANDERSON.

[Since the above was written there has appeared in *Notes and Queries* of April 19, 1941, a note by A. J. H., who had discovered an account of Thomas Bayes in the *Imperial dictionary of universal biography*, published in Glasgow in 1865.

A. J. H. gives the additional information (*a*) that the tract on "Divine Benevolence" was in reply to another anonymous tract (by Balguy) on "Divine Rectitude"; (*b*) that "Analyst" in the controversy on the calculus was none other than Bishop Berkeley the celebrated metaphysician—both of which items had been, with the aim of abbrevia-

tion, suppressed by the present writer ; and (*c*) that the dictionary had stated Bayes's death in April 1761 to have occurred on the 17th, while the contemporary *Gentleman's Magazine* gave the date as the 7th. It might have been added that the *London Magazine* pronounced it to have happened on the 14th, and it was in order to evade this uncertainty that only the month and the year have been stated above. The 17th is doubtless derived from the Bunhill fields epitaph which is probably now quite illegible. There is a manuscript transcript of it in Wilson's own (interleaved) copy—now in Dr. Williams's library—of his *Dissenting Churches* : and it shows that the epitaph was unreliable, in that Bayes's father is stated to have died at age 52, instead of about 75 ; and Wilson has pencilled a suggestion that 72 may have been intended. J. G. A.]

1521. *Laplace's Antecedent-probability function.* (Cf. Review, *post.*, p. 177).

Laplace, in his discussion of *a posteriori* probability (*Th. Anal. des Probs.*, Livre 2, Chap. VI, para. 32) takes a function z or z_x, to represent the antecedent chance or frequency of a probability equal to x : but he does not seem to have given any general form for z_x. A useful and elastic form was suggested in 1889 by the distinguished British actuary George Francis Hardy, F.I.A., afterwards Sir George F. Hardy, K.C.B. (*Insurance Record*, Vol. XXVII, p. 457, *seq.*: reproduced in *Trans. Fac. Actuaries*, Vol. VIII, No. 77 * (1920), p. 181). He remarked that the values of z_x " must be to some extent matters of opinion, and are not absolutely determinable (the only thing absolutely certain about them being that they are not equal to one another).† They will be zero when $x=0$ or 1, and rise to a maximum when x has its most probable value. Hence they will form a series which may be fairly represented by a curve of the form, $C \cdot x^r(1-x)^s$, where the relative values of r and s will depend on the most probable value of x, and their absolute values on the extent of our prior knowledge of that function." In this familiar curve we have

$$z_x = C \cdot x^r(1-x)^s$$

$$\text{Mode} = r/(r+s)$$

$$\text{Mean} = (r+1)/(r+s+2)$$

Variance, or 2nd moment $= (r+1)(s+1)/(r+s+2)^2(r+s+3)$.

The use of this form has the following advantages :

(1) By suitable choice of r and s any required values of the mode or mean and the variance of z_x can be reproduced, and thus a great variety of distributions may be approximately represented. If r

* A copy of this number of *Trans. Fac. Act.* has been placed in the Library of the Association. It deals entirely with the theory of inverse probability and includes, *inter alia*, a little known but important paper, " On some Disputed Questions of Probability ", by Professor E. T. Whittaker, F.R.S., in which he supports the theory against Chrystal's attack.

† He was dealing with chances of *death*.

and s are both positive the curve is of the familiar cocked-hat variety (symmetrical or skew as $r =$ or $\neq s$) that we may normally expect; the frequency falling to zero for the extreme values 0 and 1, and rising to a maximum at the point $r/(r+s)$, the concentration round that point—i.e. the precision of the antecedent information, being greater as r and s are increased. If $r=s=0$ the distribution is rectangular as assumed by Bayes. Less usual forms can be represented by making one or both of r and s negative (but > -1, this being the condition that the integral from 0 to 1 shall be finite). If r and s are both negative the curve will be U-shaped, starting and ending with infinite ordinates and falling to a minimum for $x=r/(r+s)$. If one of r and s is positive and the other negative the curve is J-shaped. Usually z_x will be of the cocked-hat variety, and the form $Cx^r(1-x)^s$ will represent it very fairly in the neighbourhood of the mode: this is the only part of much practical importance unless the observed probability differs widely from the mode of z_x, in which case it may be doubted whether the Laplacian formula would be of great value.

(2) The factors x^r and $(1-x)^s$ unite in the simplest and most elegant way with similar factors in the Laplacian integrand so as to produce a similar and easily integrable form, which incidentally makes it easy to exhibit and appreciate the general effect of Laplace's formula.

Suppose for example that the event whose probability is sought has happened m times and failed n times in $m+n$ trials and that $z_x=Cx^r(1-x)^s$: then Laplace's expression for the chance that the true probability is x is

$$\frac{z_x \cdot x^m(1-x)^n \cdot dx}{\int_0^1 (\text{numerator})} = \frac{x^r(1-x)^s \cdot x^m(1-x)^n \cdot dx}{\int_0^1 (\text{numerator})} = \frac{x^{m+r}(1-x)^{n+s} \cdot dx}{\int_0^1 (\text{numerator})}.$$

This is a maximum when $x=(m+r)/(m+n+r+s)$; the mean value of x is

$$p=(m+r+1)/(m+n+r+s+2).$$

Putting $r=s=0$, so that z_x is a rectangular distribution (Bayes), this reduces to the well-known form

$$(m+1)/(m+n+2).$$

Put $(m+n)=E$, the total number of new trials, also $(r+s+2)=N$ and $(r+1)/N=p'$ and $m/E=p''$, the observed value. Then the Laplacian p may be put in the form

$$p=\frac{m+r+1}{m+n+s+r+2}=p''\frac{E}{E+N}+p'\frac{N}{E+N}.$$

i.e. p is a blend of the observed value p'' and the mean antecedent value p' in the proportions E and N. Thus for any fixed number E of trials p tends more and more to p', the mean antecedent rate, as N increases, i.e. as the precision of the antecedent information increases. But for a fixed N, p tends more and more to the observed value p'' as E, the total number of trials, increases.

It is difficult to see how, in practice, z_x could be found except by a large number of antecedent trials. Suppose for example that the experiment is the throwing of a die not known to be dynamically true, and we require p the probability of throwing a six with that die. How are we to get z_x? *A priori* it might be expected that the mean would not show much departure from 1/6, but the actual departure is important but unknown, as also are the variance and the general spread of the curve. Hence it would apparently be necessary to make a large number of throws with *each* of a large number of dice drawn from the class to which the die of the main experiment belongs. In this way a frequency-table of values of the probabilities for individual dice would be obtained and an approximate frequency-curve for z_x be deduced. But it would appear that the time and effort involved would be more profitably devoted to increasing E, the total number of trials in the main experiment, thus causing the observed value p'' to approach more nearly to the true value of the probability for the individual die with which we are concerned.

It is important to remember that the Laplacian p is not a simple probability but a mean or expectation, which cannot be applied in the same way as ordinary simple *a priori* probabilities to obtain the chances of repetitions and the binomial distribution. Cf. *Trans. Fac. Act.*, Vol. VIII, section 7, pp. 173–4, and sections 10–12, pp. 185–6. In K fresh trials the expected number of successes will be pK but not with the distribution $K(p+q)^k$: and as K increases the proportion of successes will *not* tend to p, but to the unknown true probability of the event. G. J. LIDSTONE.

1522. *A property of the parabola.*

Let PSQ be a focal chord of a parabola, vertex A, and let the tangents at P and Q intersect at O on the directrix OX. Consider the circle POQ on PQ as diameter; draw the chord OR of this circle parallel to PQ, join RS and produce to cut the circle again at T.

Along PSQ mark off PW equal to SQ and let V be the mid-point of PQ. Then V is also the mid-point of SW and since OSP is a right angle, $OR = 2SV = SW$.

Again, OV, being a diameter of the parabola, is parallel to XS and so

$$AX : OX = \tfrac{1}{2}SX : OX = \tfrac{1}{2}OS : SV = OS : OR;$$

and hence $$\angle XOA = \angle ORS,$$

but also $$\angle XOT = \angle ORS,$$

since XO is the tangent at O to the circle POQ because $\angle XOV = 90°$. Thus O, A, T are collinear.

Now draw OU perpendicular to RST and TB perpendicular to OAT to cut the axis at B.

Then $$RS \,.\, ST = PS \,.\, SQ = SO^2 = SU \,.\, SR,$$

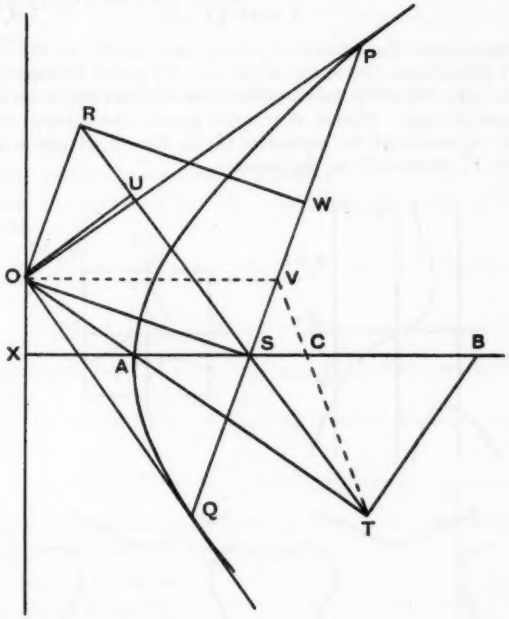
and so $$ST = SU.$$

Also, since O, X, S, U lie on a circle,

$$\angle OXU = \angle OSU = 90° - \angle ORS = 90° - \angle XOA,$$

and therefore XU is perpendicular to OAT.

It follows immediately that the triangles SUX, STB are congruent. Thus $SB = XS$, and so B is a fixed point.



If VT cuts the axis at C, $CA = CT$, since $VO = VT$.

Hence the circle on AB as diameter, centre C, always touches the circle on PQ as diameter, centre V.

The complete envelope then of circles described on focal chords of a parabola as diameters consists of the directrix together with this circle, diameter AB; and it is a remarkable fact that the line joining the two points of contact of any one such circle with the separate parts of its envelope passes through the vertex of the parabola.

E. P. Lewis.

1523. *Repeating Curves.*

In *Curve Tracing* by Percival Frost will be found a section on the tracing of curves such as those represented by

$$\tan y + \sin^3 x + \sin x \tan^2 y (\tan y + \sin^2 x) = 0. \quad \ldots\ldots\ldots\text{I}$$

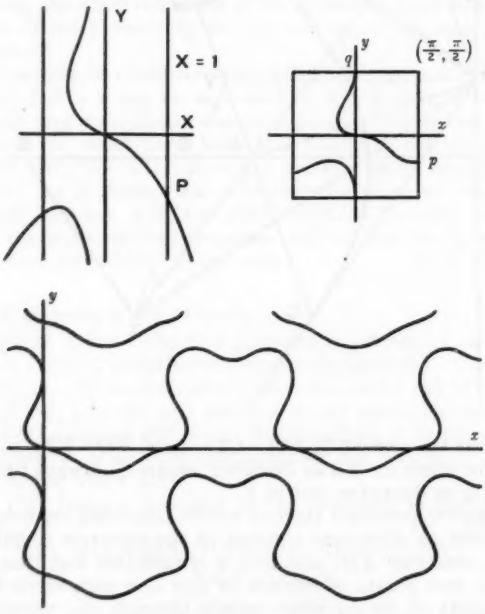
There an ingenious construction is used to connect (I) with

$$Y + X^3 + XY^2(Y + X^2) = 0. \quad \text{.....II}$$

From another point of view the subject could be used to introduce sixth formers to the idea of mapping.

Consider $\left.\begin{aligned} X &= \sin x \\ Y &= \tan y \end{aligned}\right\}$ $\dfrac{dy}{dx} = \cos x \cos^2 y \dfrac{dY}{dX}$.

This transforms the square of side π and centre at the origin in the (x, y) plane into the strip of the (X, Y) plane bounded by the lines $X = \pm 1$. At all points inside these regions dy/dx and dY/dX have the same sign. Hence at interior points the general shapes of the curve represented by equation (I) in the (x, y) plane and (II) in the (X, Y) plane will be the same.



We note that, at p, dy/dx is zero; to investigate the shape of the curve at q let $\xi = \frac{1}{2}\pi - y$.

Using the fact that for small X and large Y $X \simeq -1/Y^2$, we get $x \simeq -\xi^2$ for small x; the shape at $(0, -\frac{1}{2}\pi)$ is determined similarly.

It is easily verified that $X = -1$ cuts (II) in one real point only.

Thus the shape of (I) in the square is determined and considerations of periodicity result in the pattern shown over the (x, y) plane.

By varying the equation chosen or by using two sines or two tangents, many varieties of pattern can be obtained.

J. H. CADWELL.

1524. *Proof of an identity.*

$$\left.\begin{aligned}\sum_{r=0}^{x}\binom{t}{r}(-p+s)_r\binom{s+t-r}{x-r}(p)_{x-r}&=\binom{s}{x}(p+t)_x \quad (x\leqq s)\\ &=0 \qquad\qquad (x>s)\end{aligned}\right\}. \quad \ldots(\mathrm{I})$$

In the above identity t, s, r and x are positive integers, $x \leqslant s+t$, and $\binom{t}{0}=1$.

1. *Proof using Vandermonde's Theorem.*

Writing $(n)_t=n(n-1)(n-2)\ldots$ to t factors, where $(n)_0=1$, we know that

$$(n)_t=\sum_{r=0}^{t}\binom{t}{r}(-p+s)_r(n+p-s)_{t-r}, \text{ (Vandermonde) (II)}$$

also
$$(n+p)_s=\sum_{x=0}^{s}\binom{s}{x}(p+t)_x(n-t)_{s-x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{III})$$

Thus multiplying II by $(n+p)_s$ and III by $(n)_t$, we have

$$(n)_t(n+p)_s=\sum_{r=0}^{t}\binom{t}{r}(-p+s)_r(n+p)_{s+t-r}, \quad \ldots\ldots\ldots(\mathrm{IV})$$

and
$$(n+p)_s(n)_t=\sum_{x=0}^{s}\binom{s}{x}(p+t)_x(n)_{s+t-x}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(\mathrm{V})$$

We now equate the coefficients of $(n)_{s+t-x}$ on the R.H. sides of (IV) and (V), and obtain the result (I).

2. *A proof by Induction.*

When $x=1$, (I) gives

$$(s+t)p+t(-p+s)=s(p+t) \quad \text{which is true.}$$

Let us assume (I) is true for some value x; and multiply each side by $\dfrac{(s-x)(p-x+t)}{x+1}$.

We multiply the rth term (T_r) on L.H.S. by X_r+Y_r, where

$$X_r=\frac{(p-x+r-1)(s+t-x-r+1)}{x+1}, \quad Y_r=\frac{(t-r+1)(s-p-r+1)}{x+1},$$

$$X_r+Y_r=\frac{(s-x)(p-x+t)}{x+1}.$$

Then the general term in the product is

$$X_{r+1}T_{r+1}+Y_rT_r=\binom{t}{r}(-p+s)_r\binom{s+t-r}{x+1-r}(p)_{x+1-r},$$

and so we have

$$\sum_{r=0}^{x+1}\binom{t}{r}(-p+s)_r\binom{s+t-r}{x+1-r}(p)_{x+1-r}=\binom{s}{x+1}(p+t)_{x+1} \quad (x<s)$$

$$=0 \quad (x\gtrless s).$$

Hence the result (I) is true for $x+1$.
This completes the proof. H. J. CURNOW.

1525. *An exercise in symmetry.*

The old-fashioned permutations problems, to be found in almost every algebra book, about seating people at round tables and arranging beads on strings have their chemical counterpart in the problem of finding the number of substitution products that can be formed from a molecule of benzene. For more elaborate compounds, such as anthracene ($C_{14}H_{10}$), the problem provides a good exercise in symmetry.

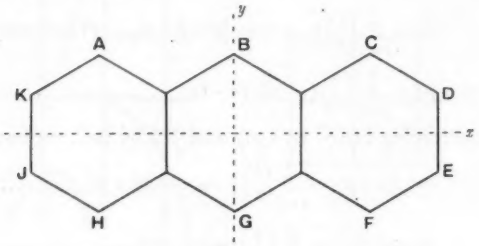


FIG. 1.

We first consider all ten hydrogen atoms to be distinct ; that is to say, we liken the molecule to a dining-table, with the places numbered and reserved, rather than to a wire framework that can be turned round or turned over. We can then choose two positions out of the ten in ${}_{10}C_2$, that is, in 45 ways. Of these, one pair (*BG* in the figure) is symmetrical about both the axes shown ; four more (*KJ*, *AH*, *CF*, *DE*) are symmetrical about the x-axis only, four more about the y-axis ; four more about the z-axis (central symmetry) ; and the remaining 32 are unsymmetrical. If now the figure is rotated through 180° about each of the three axes in turn, it is seen that the 32 unsymmetrical pairs form 8 groups of four, the 12 singly-symmetrical pairs form 6 groups of two, and the doubly-symmetrical pair forms one group by itself ; each of these 15 groups representing a distinct arrangement for a framework that can be turned in any manner. There are thus 15 di-substitution products of anthracene when the two substitution groups are identical (and 30 if they are distinct).

For a tri-substitution product, there is no doubly-symmetrical choice ; there are 8 singly-symmetrical trios ; and of the ${}_{10}C_3$ (that

is, 120) possible choices, the remaining 112 are therefore unsymmetrical. The number of tri-substitution products (for identical substitution groups) is therefore $\frac{1}{4} . 112 + \frac{1}{2} . 8$, that is, 32.

For a quadri-substitution product, the ${}_{10}C_4$ (that is, 210) combinations include two which are doubly-symmetrical, 24 singly-symmetrical, and hence 184 unsymmetrical. The number of such products (for identical substitution-groups) is therefore $\frac{1}{4} . 184 + \frac{1}{2} . 24 + 2$, that is, 60.

On the same lines it is possible to construct general formulae. Thus for a compound represented by n hexagons ($n>1$) joined together in a straight line the number of $2r$-substitution products (the substitution-groups being identical) would be

$$\tfrac{1}{4}({}_{2n+4}C_{2r}) + \tfrac{3}{4}({}_{n+2}C_r) ;$$

and the number of $(2r+1)$-substitution products would be $\frac{1}{4}({}_{2n+4}C_{2r+1})$ if n is even, or $\frac{1}{4}({}_{2n+4}C_{2r+1}) + \frac{1}{2}({}_{n+1}C_r)$ if n is odd.

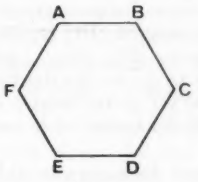


FIG. 2.

The exceptional case $n=1$ is not without interest. The unsymmetrical trio ABD is one of a group of twelve, and though these can be related to each other by rotations of 180° about the various axes of symmetry, it is a good deal simpler to ignore five out of the six axes of symmetry in the plane and to relate the twelve positions to each other by rotations of 60° about the z-axis, together with a rotation of 180° about any one of the axes in the plane. For a combination which is itself symmetrical about an axis in the plane, this last rotation is not needed. Thus the pair AB is one of a group of six ; the pair AD, itself symmetrical about the z-axis, is one of a group of three ; while the trio ACE, which can be rotated into itself by rotations of 120°, is one of a group of two.

For a di-substitution product, a pair can be chosen in ${}_6C_2$, that is, 15 ways. The 15 pairs consist of two groups of six and one group of three. There are thus three possibilities. For a tri-substitution product a trio can be chosen in ${}_6C_3$, that is, 20 ways, making one group of twelve, one group of six and one group of two. There are therefore three possibilities.

These last considerations suggest that symmetry may profitably be thought of in terms of rotation. The group ACE might be regarded as having " three-fold cyclic symmetry " about the z-axis.

Cyclic symmetry is a useful conception in algebra: why not also in geometry? The feature so vividly exhibited by the coat-of-arms of the Isle of Man ought surely to be given some place in mathematics.

E. H. LOCKWOOD.

1526. *A first lesson on kinematics. Application of graphs.*

The following lesson proved particularly interesting and may give a useful suggestion to any who are anxious to broaden the scope of their work on graphs. It was given to an intelligent class of boys, mostly under 15, who have been drawing graphs of functions. It is a term in which I give some first ideas on dynamics —preparatory to a fuller course next year. This is the first lesson, of three-quarters of an hour.

All the boys had squared paper, and pencils, and were asked to draw the graphs I describe freehand. I suggested easy scales —our paper has four squares to an inch—and chose easy numbers. I have not reproduced the graphs here—to save unnecessary printing costs. [Here speaks the treasurer.]

The first graph is a time-distance graph. Time is to be represented horizontally, showing 5, 10, 15, 20, 25 min. on the axis; distance vertically showing 200, 400, ... 1000 yd. on the axis. A boy sets off from home to go to the dentist; he walks 300 yd. in the first 5 min., runs 600 yd. in the next 5 min., hesitates and stops for 5 min., and then thinks better of it and runs home at 150 yd. per min.

Points raised in a short discussion on this were that the speed is represented by the gradient, and that the "return" involved only coming back on the distance scale. Several wanted to make their watches go backwards also. But the ideas here were familiar so we did not bore the brighter boys by stopping long on this graph.

Next represent this same journey on a time-velocity graph, taking the same scale for time, and 40, 80, 120 yd. per min. vertically. This proved most interesting. Comparatively few at first saw that the graph consisted of lines parallel to the time-axis. It gave them a real example of a discontinuous graph. They needed coaxing before realising that the return journey could be represented by a negative velocity—one of their first introductions to the idea of "sense" in measuring a line.

Now for the reverse process, i.e. a time-velocity graph drawn first and the time-distance graph deduced. Taking the same scale for the time-velocity graph, represent the following journey. A boy starts at 80 yd. per min. for 5 min., and then increases his speed uniformly, i.e. "accelerates" uniformly, for the next 5 min. to 160 yd. per min. He stops for 5 min. and then goes back at a speed which is at first 100 yd. per min., and decreases uniformly to 80 yd. per min. in the course of the next ten minutes. [Nothing was said about where this landed the boy.]

After I had sketched this on the board, and the boys had altered their graphs where wrong—most of them were correct—I asked

them to draw the corresponding time-distance graph, using the same scale as before. This proved exciting, and walking round the class I was able to see the progress made. The first 5 min. was done correctly, but the next 5 min. was baffling. On my first round I announced that no one was correct; that stirred them and three boys then drew a curve correctly, and giving hints I got half the class to get it right before I drew it on the board. Then the question was raised as to where the boy was at the end of the 25 minutes. Working from the time-velocity graph this introduced the idea of average velocity, and area under the graph, and the distance was calculated, and boys had a check on the accuracy of their freehand time-distance graph.

They had in these graphs met the idea of the gradient of the tangent to a curve representing velocity, as well as of the area under a graph representing distance. I had just time to draw a curved graph, time-velocity, given as speedometer readings of a car at various times, to show the importance of this latter idea for "irregular" motion. K. S. SNELL.

1527. *The Tail of Stirling's Formula for log n!*

The tail of Stirling's formula consists of the terms

$$1/12n - 1/360n^3 + 1/1260n^5 - 1/1680n^7 + 1/1188n^9 - \dots .$$

The first two of these are reproduced by the first two in the expansion of $A \sin \theta/n$, with $A = \sqrt{5}/12$ and $\theta = \sqrt{1/5}$. For logs. to the base 10 we multiply by the modulus and get the formula

$$\cdot 080929 \sin (25^\circ \cdot 623/n)$$

attributed to Karl Pearson in *Tables for Statisticians ...*; Part I, p. lv. This requires reference to trigonometric tables, and a conversion of the angle to the sexagesimal system if a table with decimal arguments is not available.

An alternative formula devised by the writer was published in *Nature*, of August 25, 1923, Vol. CXII, p. 283, where it now lies decently buried: it may be useful to dig it up and re-inter it in the pages of the *Gazette*. The first three terms of the tail are exactly reproduced by the first three of the expansion of the binomial

$$\frac{1}{12n}\left(1 + \frac{113}{210n^2}\right)^{-7/113},$$

while the simpler binomial

$$\frac{1}{12n}\left(1 + \frac{8}{15n^2}\right)^{-1/16} \text{ or } \frac{1}{12n}\left(\frac{15n^2}{15n^2+8}\right)^{1/16}$$

reproduces exactly the terms in $1/n$ and $1/n^3$ and with very close approximation the term in $1/n^5$. Adopting this simpler form and bringing in the principal terms we get

$$\log_e n! \doteqdot \log_e \sqrt{2\pi} + (n + \tfrac{1}{2}) \log_e n - n + \frac{1}{12n}\left(\frac{15n^2}{15n^2+8}\right)^{1/16}.$$

Passing to common logarithms we get

$$\log_{10} n! \doteqdot 0{\cdot}39908993 \ldots + (n+\tfrac{1}{2}) \log_{10} n - nM + \frac{M}{12n}\left(\frac{15n^2}{15n^2+8}\right)^{1/16},$$

where M is the modulus, ·4342944819 .... This gives for 1! (exact value 1), 1·00007 ... ; for 2! (true value 2), 2·000002 ; for 3! and 5! no discrepancy was found by 7-figure logs. and 9-figure logs. respectively ; and for $n = 10$ the log. is correct to 10 places. The error in the common log. (approx. value less true value) is

$$M[-1/151200n^5 + 1391/4536000n^7 \ldots]$$
$$= M[-30 + 1391/n^2 \ldots]/4536000n^5.$$

Up to about $n = 7$ the second term of the error predominates and the error is positive ; thereafter the first term predominates and the error is negative but rapidly becomes quite insignificant, and the formula virtually exact.

The high degree of approximation for even small values of n makes the formula specially useful when n is not integral, i.e. the value of $\log \Gamma(n+1)$ is required for values of n which are not great. This problem often arises in curve-fitting by the Pearsonian system of frequency-curves. The alternative method of reduction by factors is not very convenient when n is at all considerable, especially of the log. of each factor involves the use of proportional parts. In any case it is useful to have, as a check, an independent and self-contained formula not involving reference to a table of $\log \Gamma(x)$.

Some other interesting approximate formulae are given and illustrated by Mr. James Henderson, *Nature*, Vol. CXII, pp. 96–8, but they do not appear to be as easily derived, or as simple, or as accurate as the one above referred to. G. J. LIDSTONE.

1528. *The integral* $\int (1 + e\cos\theta)^{-n}\, d\theta$.

This Note gives a geometrical investigation of the well-known substitutions

$$(1 + e\cos\theta)(1 - e\cos\phi) = (1 - e^2);$$
$$(1 + e\cos\theta)(e\,\mathrm{ch}\,\phi - 1) = (e^2 - 1).$$

1. $\int (1 + e\cos\theta)^{-n}\, d\theta$, where $|e| < 1$.

Consider a point P on an ellipse. We have

the polar equation $l/r = 1 + e\cos\theta$; .....(i)

the focus property $r = a(1 - e\cos\phi)$; .....(ii)

and the ordinate property $r\sin\theta = b\sin\phi$, ...(iii)

where ϕ is the eccentric angle of P.

From (i) and (ii), $(1+e\cos\theta)(1-e\cos\phi)=(1-e^2)$.

Also, by differentiating these two equations,

$$l\,dr=er^2\sin\theta\,d\theta,\quad dr=ae\sin\phi\,d\phi,$$

so, dividing one of these by the other,

$$\frac{d\theta}{d\phi}=\frac{al}{r^2}\cdot\frac{\sin\phi}{\sin\theta}=\frac{al}{br}$$

$$=\frac{a}{b}(1+e\cos\theta),\quad \text{using (i) and (iii).}$$

Hence
$$\int\frac{d\theta}{(1+e\cos\theta)^n}=\frac{a}{b}\int\frac{d\phi}{(1+e\cos\theta)^{n-1}}$$

$$=\frac{a}{b}\int\frac{(1-e\cos\phi)^{n-1}}{(1-e^2)^{n-1}}\cdot d\phi$$

$$=(1-e^2)^{\frac{1}{2}-n}\int(1-e\cos\phi)^{n-1}\,d\phi.$$

In simple cases the limits of ϕ corresponding to those of θ are obvious from the ellipse.

Thus
$$\int_0^{2\pi}d\theta/(1+e\cos\theta)=\int_0^{2\pi}(r/l)(la/rb)\,d\phi$$

$$=\int_0^{2\pi}(a/b)\,d\phi=2\pi/\sqrt{(1-e^2)}.$$

Also
$$\int_0^{2\pi}d\theta/(1+e\cos\theta)^2=(2/l^2)\int_0^{2\pi}\tfrac{1}{2}r^2\,d\theta$$

$$=2\pi ab/l^2=2\pi/\sqrt{(1-e^2)^3}.$$

Sometimes it is possible to use r as the new variable, for

$$\sin\theta\,d\theta=l\,dr/er^2,$$

and so $\int_{\pi/2}^{\alpha}\sin^5\theta\cos^3\theta(1+e\cos\theta)^{-9}\,d\theta$, where $\alpha=\pi-\cos^{-1}e$

is only
$$(1/e^8l^8)\int_l^a(l-r)^3\{e^2r^2-(l-r)^2\}^2\,dr,$$

the point P in the ellipse being taken from the end of the latus rectum to the end of the minor axis, in the first quadrant.

2. $\int(1+e\cos\theta)^{-n}\,d\theta$, where $|e|>1$.

Consider a point P on a hyperbola. As in 1, we have three equations :

$$l/r=1+e\cos\theta\,;\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i')}$$

$$r=a(e\operatorname{ch}\phi-1)\,;\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii')}$$

$$r\sin\theta=b\operatorname{sh}\phi,\quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(iii')}$$

where ϕ is the parameter of the point P.

From (i') and (ii'), $(1+e\cos\theta)(e\operatorname{ch}\phi-1)=(e^2-1)$.

Also, by differentiation, we have

$$l\,dr = er^2 \sin\theta\,d\theta, \quad dr = ae\,\mathrm{sh}\,\phi\,d\phi,$$

whence, dividing,

$$\frac{d\theta}{d\phi} = \frac{a}{b}\cdot\frac{l}{r} = \frac{a}{b}(1+e\cos\theta), \quad \text{using (i') and (iii').}$$

Hence

$$\int\frac{d\theta}{(1+e\cos\theta)^n} = \frac{a}{b}\int\frac{d\phi}{(1+e\cos\theta)^{n-1}}$$

$$= \frac{a}{b}\int\frac{(e\,\mathrm{ch}\,\phi-1)^{n-1}}{(e^2-1)^{n-1}}\cdot d\phi$$

$$= (e^2-1)^{\frac{1}{2}-n}\int(e\,\mathrm{ch}\,\phi-1)^{n-1}\,d\phi.$$

C. BOND.

1529. *The non-convergence of sin* $(n\alpha+\beta)$.

It is well known, though not always simple to prove, that *sin* $(n\alpha+\beta)$ *does not converge as* $n\to\infty$, except for trivial cases covered by $\sin\alpha=0$. The following short proof may be new.

We know that, if u_n converges, then $u_{n+1}-u_{n-1}$ converges to zero. Suppose that $\sin\alpha\neq0$ and that $\sin(n\alpha+\beta)$ converges. Then, taking $u_n\equiv\sin(n\alpha+\beta)$, we get $\cos(n\alpha+\beta)\to0$. We can thus similarly take $u_n\equiv\cos(n\alpha+\beta)$, which gives $\sin(n\alpha+\beta)\to0$. We have therefore reached the contradiction

$$1=\cos^2(n\alpha+\beta)+\sin^2(n\alpha+\beta)\to0,$$

and so, if $\sin\alpha\neq0$, $\sin(n\alpha+\beta)$ does not converge.

T. W. CHAUNDY.

1530. *An exercise in limits.*

To find the radius of curvature of $x^3+y^3=3axy$ at the origin.

1. $dy/dx=(ay-x^2)/(y^2-ax)$ which is indeterminate at (0, 0).

2. Putting $y=tx$ we obtain the parametric representation

$$x=3at/(1+t^3), \quad y=3at^2/(1+t^3).$$

Using this and the usual formula we find, after some reduction,

$$\rho=3a/2.$$

3. Noting that the curve touches the axes at the origin we might try Newton's method. The equation can be written

$$(x^2/y)+(y^2/x)=3a,$$

whence

$$\lim(x^2/y)+\lim(y^2/x)=3a, \qquad \text{.....(i)}$$

or, with an obvious notation,

$$2\rho_x+2\rho_y=3a.$$

Since $\rho_x = \rho_y$ by symmetry, we find

$$\rho_x = \rho_y = 3a/4,$$

which is not correct.

The error lies in the fact that $\lim(x^2/y)$ has been evaluated on the branch touching the axis of x, while $\lim(y^2/x)$ has been evaluated on the branch touching the axis of y. Considering only the branch touching the axis of x,

$$y = (x^3 + y^3)/3ax$$
$$= (x^2/3a) + (y^3/3ax),$$

and the equation can be written

$$y = (x^2/3a) + \ldots .$$

Using this in (i),

$$\lim_{x\to 0}\left(\frac{x^2}{y}\right) + \lim_{x\to 0}\left[\left(\frac{x^4}{9a^2} + \ldots\right)\Big/x\right] = 3a.$$

and so

$$2\rho_x + 0 = 3a.$$

We thus obtain the correct values

$$\rho_x = 3a/2 = \rho_y.$$

J. PEDOE.

1531. *Transformations of Cartesian Equations.*

Talks with a number of my pupils, all of whom have obtained a Higher Certificate and some of whom are State Scholars, have revealed an unexpected gap in their knowledge of coordinate geometry, and as it is possible (unless my selection of pupils is far from random) that this gap is widespread, I have drafted this short Note in the hope of attracting the attention of teachers. I have not noted any textbook where the point is set out explicitly, though it is of course implicit in most of them.

If (x, y) are the cartesian coordinates of a point, we consider the transformation to coordinates (ξ, η) by means of the equations

$$\rho\xi = px + qy + a,$$
$$\rho\eta = qy - px + b.$$

The new axes $\xi = 0$, $\eta = 0$ are perpendicular, and so are a possible pair of axes for cartesian coordinates. The point which has caused trouble is that the factor ρ (which each pupil omitted in the question from which this problem arose) is uniquely determined if the calculated lengths of a line, measured by $\sqrt{\{(x_1 - x_2)^2 + (y_1 - y_2)^2\}}$ or $\sqrt{\{(\xi_1 - \xi_2)^2 + (\eta_1 - \eta_2)^2\}}$, are to be the same in each system of coordinates. An equivalent property, simpler for the determination of ρ, is that the distance between the lines $\xi = \alpha$, $\xi = \beta$ must be precisely $|\alpha - \beta|$, with a similar result for lines $\eta =$ constant.

Now if O is the origin $x = y = 0$, the length of the perpendicular from O on to the line $\xi = \alpha$, i.e. $px + qy + a - \rho\alpha = 0$, is well known to be $(px + qy + a - \rho\alpha)/\sqrt{(p^2 + q^2)}$, with a similar result for $\xi = \beta$. The

distance between these lines is (numerically, if $\alpha>\beta$, with suitable adjustments in other cases) $\rho(\alpha-\beta)/\sqrt{(p^2+q^2)}$, which is equal to $\alpha-\beta$ if $\rho=\sqrt{(p^2+q^2)}$.

All this is, of course, well known to teachers, but the point that I wish to make is that their pupils seem to be missing it. I add two examples of stock type to illustrate where the precise application is wanted, choosing them as simply as possible :

1. Find the latus rectum of the parabola

$$x^2+4xy+4y^2-6x+8y-11=0.$$

The equation is

$$(x+2y+1)^2=4(2x-y+3).$$

(My pupils would have deduced $l=4$.)

Let

$$\xi\sqrt{5}=x+2y+1,$$

$$\eta\sqrt{5}=2x-y+3,$$

so that

$$5\xi^2=4\eta\sqrt{5},$$

showing that $l=4/\sqrt{5}$.

2. Find the radii of curvature at the origin of the curve $x^2-y^2=x^3$.

The curve has two branches at the origin, the tangents being $x\pm y=0$.

Let

$$\xi\sqrt{2}=x+y$$

$$\eta\sqrt{2}=x-y,$$

so that

$$x=(\xi+\eta)/\sqrt{2}.$$

The equation is $2\xi\eta=(\xi^3+3\xi^2\eta+3\xi\eta^2+\eta^3)/2\sqrt{2}$,

so that

$$\frac{\xi^2}{2\eta}+\frac{3}{2}\xi+\frac{3}{2}\eta+\frac{\eta^2}{2\xi}=2\sqrt{2}.$$

Hence, for the branch touching $\xi=0$,

$$\rho=\lim(\eta^2/2\xi)=2\sqrt{2},$$

and, for the branch touching $\eta=0$,

$$\rho=\lim(\xi^2/2\eta)=2\sqrt{2},$$

the limiting process being assumed familiar. E. A. MAXWELL.

1370. There are no eternal theories in science. It always happens that some of the facts predicted by a theory are disproved by experiment. Every theory has its period of gradual development and triumph, after which it may experience a rapid decline. . . . Nearly every great advance in science arises from a crisis in the old theory, through an endeavour to find a way out of the difficulties created. We must examine old ideas, old theories, although they belong to the past, for this is the only way to understand the importance of the new ones and the extent of their validity.—Albert Einstein and Leopold Infeld, *The Evolution of Physics* (Cambridge), 1938, pp. 77–8. [Per Mr. Frank Robbins.]

REVIEWS.

Facsimiles of Two Papers by Bayes. (i) An Essay towards solving a Problem in the Doctrine of Chances, with Richard Price's Foreword and Discussion; *Phil. Trans. Roy. Soc.*, pp. 370-418, 1763. With a Commentary by Edward C. Molina. (ii) A Letter on Asymptotic Series from Bayes to John Canton; pp. 269-271 of the same Volume. With a Commentary by W. Edwards Deming. Pp. 52+xvi; $8\frac{1}{2}'' \times 5\frac{1}{2}''$. $1. 1940. (The Graduate School, The Department of Agriculture, Washington)

This slender, attractive and well-produced volume should be in every mathematical and statistical library. As remarked by Mr. Molina (of the Bell Telephone Laboratory) in his introductory comments on (i): "No part of probability has been discussed with so much heat, before or since 1763 [as the theory of inverse or inductive probability.]..... No contribution in this line of thought has been so much quoted as Bayes' essay, yet only a privileged few have had the opportunity of seeing just what Bayes did. The purpose of this publication is to lift the essay from the obscurity of a few extant volumes of the Transactions of the Royal Society." The great majority of students, and perhaps many specialists and critics, have had to be content with the valuable but necessarily much condensed account of the Essay in Todhunter's *History of the Theory of Probability*, itself out of print and relatively inaccessible.* But this is particularly a case in which reference to the original source and treatment is highly enlightening and virtually essential. It was therefore a happy idea and a valuable service to re-publish the Essay (as communicated by Price with commentary and notes), and to do so in facsimile (full size but with reduced margins). Since facsimile reproduction does not admit of the correction of errors, Dr. Deming in the Editorial Preface records those that have been so far detected, including two mentioned by Price in his second paper (*Proc. Roy. Soc.*, LIV, p. 296, not reprinted in this collection). One of these errors considerably affects one of the numerical examples given in Price's Appendix, and shows that Bayes' Second Rule for computation is in that case considerably more accurate than appears from the example as printed. Dr. Deming discusses Bayes' use of the rather pleasing plural form "serieses", and shows that it is not a misprint since the same form was used by James Gregory in 1671. We may point out that Price in different places uses the variant plural forms "series", "series's", and "serieses", and that these three forms are given in the *O.E.D.* with other examples.

It is curious that little seems to be known about the author of such a famous essay†. His name does not appear in the *D.N.B.*, or in the *Encycl. Brit.*, or in the Biographical Section of the *English Cycl.*, with which De Morgan was so closely associated; and the reviewer has tried unsuccessfully to obtain information from the Royal Society. It is thought that readers of the essay will arise from its perusal with the definite impression that Bayes had a clear and penetrating mind with considerable power of algebraic manipulation, orderly arrangement and clear exposition; and it is not surprising that Price should have remarked: "He had, you know, the honour of being a member of that illustrious Society [i.e. the Royal Society], and was much esteemed by many in it as a very able mathematician."

A good example of Bayes' penetrative power is contained in his little-known letter printed second in the present volume but actually published before the

* Note that Todhunter's *History* has been reproduced photographically in the U.S.A.: New York, G. E. Stechert & Co., 1931.

† [See, however, a biographical Note by J. G. Anderson, M.A., *ante*, p. 160. Ed.]

Essay. In this letter he considers Stirling's formula for $\log_e z$, viz. as he says $\frac{1}{2}\log_e(2\pi)+(z+\frac{1}{2})\log_e z$ lessened by the series

$$z-\frac{1}{12z}+\frac{1}{360z^3}-\frac{1}{1260z^5}+\frac{1}{1680z^7}-\frac{1}{1188z^9}+\text{ etc.}$$

He shows a very clear recognition (perhaps, as Dr. Deming remarks, "the first recognition) of the existence of an optimum term, i.e. of the asymptotic character of the series." Bayes says: "And it is true that this expression will very nearly approach to the value of that sum [i.e. the sum of the logs. of 1, 2 ... z] when z is large, and you take in only a proper number of the first terms of the foregoing series: but the whole series can never properly express any quantity at all; because after the 5th term the coefficients begin to increase, and they afterwards increase at a greater rate than what can be compensated by the increase of the powers of z, though z represent a number ever so large ... and therefore the whole series can have no ultimate value whatsoever." In his proof of this he gives an ingenious recurrence relation (new to the reviewer) between the coefficients 1/12, 1/360, 1/1260, etc. "Take $a=1/12$, $5b=a^2$, $7c=2ba$, $9d=2ca+b^2$, $11e=2da+2cb$ ... and so on; then take $A=a$, $B=2b$, $C=2\times3\times4c$, $D=2\times3\times4\times5\times6d$ ... and so on, and A, B, C, D, E, F, etc., will be the coefficients of the foregoing series: from whence it easily follows that if any term in the series after the 3 first be called y, and its distance from the first term n, the next term immediately following will be greater than

$$\frac{n\times(2n-1)}{6n+9}\times\frac{y}{z^2}.$$

Wherefore at length the subsequent terms of this series are greater than the preceding ones, and increase in infinitum." Bayes points out that other series which he mentions are of the same asymptotic character.

Dr. Deming points out [see Molina's notes, p. xi] that Bayes virtually anticipated an important purely analytical result of Laplace's. In obtaining an area which is equal to $\int_0^x x^p(1-x)^q\,dx$, Bayes expands $(1-x)^q$ by the binomial theorem, and integrates term by term (which is legitimate since the series terminates), obtaining the result

$$\text{(A)}\qquad \int_0^x x^p(1-x)^q\,dx=\frac{x^{p+1}}{p+1}-q\frac{x^{p+2}}{p+2}+\frac{q(q-1)}{2}\frac{x^{p+3}}{p+3}-\text{, etc.}$$

He then states a by no means obvious transformation of the r.h.s., obtaining the result ($r\equiv1-x$).

$$\text{(B)}\quad \int_0^x x^p(1-x)^q\,dx=\frac{x^{p+1}r^q}{p+1}+\frac{q}{p+1}\cdot\frac{x^{p+2}r^{q-1}}{p+2}+\frac{q(q-1)}{(p+1)(p+2)}\cdot\frac{x^{p+3}r^{q-2}}{p+3}+\text{etc.}$$

This transformation he says "will easily be seen by putting in the former [i.e. (A)] instead of r its value $1-x$, and expanding the terms and ordering them according to the powers of x. Or, more readily, by comparing the fluxions of the two series, and in the former [i.e. (B)] instead of $\dot{r}$ substituting $-\dot{x}$"; the latter proof is given *in extenso* by Price in a footnote. Now if we multiply the l.h.s. of (B) by the reciprocal of $\int_0^1 x^p(1-x)^q\,dx$ and the r.h.s. by its equivalent $(p+q+1)!/p!\,q!$, we get as Deming points out

$$\text{(C)}\quad \int_0^x \div\int_0^1=\frac{(p+q+1)!}{(p+1)!\,q!}x^{p+1}r^q+\frac{(p+q+1)!}{(p+2)!(q-1)!}x^{p+2}r^{q-1}+\text{etc.}$$
$$=\text{the last }(q+1)\text{ terms of }(x+r)^{p+q+1}=(x+\overline{1-x})^{p+q+1}.$$

Thus Bayes uses his result to evaluate the integral as the sum of a number of terms in the binomial expansion, multiplied by a constant factor; while his great successor Laplace—who found other means of evaluating the integral for large values of the exponents—used (C) for the converse purpose of summing a large number of terms of the binomial.

Bayes prefaced his main proposition by a brief treatment of the general laws of chance. Todhunter (*History ...*, p. 295) says that his treatment is obscure and compares very unfavourably with that of De Moivre. This judgment seems rather harsh,* and it is in any case based on a particularly high standard of comparison. Bayes, who can hardly have been ignorant of De Moivre's work, at any rate had his own reasons for a different treatment. Reversing the usual order of things he takes the idea of expectation as primary and that of probability as derivative. Dealing as he was with the new and rather subtle conception of the chance of a probability, he seems to have thought that the generally understood idea of expectation would form the simplest basis of argument for his preliminary propositions. See Price's remarks, p. [375].

We shall not plunge into the vast and stormy sea of controversy that has raged round Bayes' formula, but we may emphasize that he had no part or lot in Laplace's much-discussed formula $(m+1)/(m+n+2)$. This is not really the value of the *a posteriori* probability of the event—which is undiscoverable—but a kind of mean or expectation of that probability, which cannot be used in the same way as a simple probability [see Mathematical Note, No. 1521, *ante*, p. 162]. Bayes had no such conception. What he obtained, on the assumption of equal *a priori* chances or the "equal distribution of ignorance", was the chance that the true probability lies between given limits: or as he quaintly expresses it, page [400]:

"If nothing is known concerning an event but that it has happened p times and failed q in $p+q$ or n trials, and from hence I guess that the probability of its happening at a single trial lies between $p/n+z$ and $p/n-z$; ... my chance to be in the right is greater than [ ] and less than [ ]."

This order of ideas is quite different from that of Laplace's formula. It appears from Price's notes that Bayes' object was to substitute something with a surer logical foundation for Bernoulli's and De Moivre's use of the debatable inverse form of Bernoulli's Theorem: and so to confirm and control "*analogical* or *inductive reasoning*", as to which Price says, "it is, in all probability, the source of many errors, which perhaps might in some measure be avoided, if the force that this sort of reasoning ought to have with us were more distinctly and clearly understood." Price gives—was he the first to give?—the now notorious example of "a person just brought into this world", who (after the considerable observation-period of nearly 3000 years!) has seen the sun "return at regular and stated intervals a million of times". Price gives, according to Bayes' rules, the probability ·5352 that to this man the chance of the sun's rising again is not *greater* than 1,600,000 to one, and ·5105 that it is not *less* than 1,400,000 to one. But Price recognises more clearly than some of his distinguished successors the distinction between *casual* and *causal*; for he adds, "It should be carefully remembered that

* De Morgan, who was no bad judge, was much more appreciative. In *Notes and Queries*, 2nd Ser., Vol. IX (7 Jan. 1860) he said: "Bayes shows a very superior mathematical power: and Laplace, who makes but slight mention of him, is very much indebted to him. ... Accordingly, Bayes is one of the chief leaders in the mathematical theory of probabilities. What he did was of small extent, judged by paper and print, but of fundamental importance and wide consequence: he is of the calibre of De Moivre and Laplace in his power over the subject."

these deductions suppose a previous total ignorance of nature. After having observed for some time the course of events it would be found that the operations of nature are in general regular, and that the powers and laws which prevail in it are stable and permanent."

Altogether, this stimulating little work is to be highly commended. It should be emphasised that it does not include Price's second paper [*Phil. Trans.*, LIV, p. 296 (1764)], in which he gave Bayes' proof of his second Rule of approximation, with some additions and improvements of his own.

G. J. LIDSTONE.

Relaxation Methods in Engineering Science: A Treatise on Approximate Computation. By R. V. SOUTHWELL. Pp. viii, 252. 17s. 6d. 1940. (Oxford University Press)

Is not successive approximation the most fundamental of mathematical processes? * Southwell's "Relaxation Method" is nothing but successive approximation—but with a *physical background* which both suggests and guides the process. This physical background should be the method's first recommendation to those (engaged upon engineering design problems) for whom it was primarily intended. Even more important, however, is the development of a *technique*, especially since this can be carried over to problems where the physical background is hazy, or non-existent. But to begin at the beginning.

Many structures (aeroplane frames, structural steelwork) are highly redundant, and exact analysis leads to a number of simultaneous linear equations equal to the order of redundancy. In the case of the redundant pin-jointed frame (the first that Southwell treated, and the one from which the terminology of the method is derived) the equations imply minimum strain energy. It is a matter for congratulation that Southwell was (apparently) unaware of the Gauss-Seidel attack upon the (precisely equivalent) systems of normal equations arising in the method of least squares. Instead of formulating and solving (even approximately) these equations, Southwell visualised a process whereby the load was taken piecemeal by the structure, a process which could indeed actually be carried through, at least with a model in the laboratory. The unstrained frame has *all* its joints pinned to a rigid support (nailed to a wall). The loads are now applied, but are at first all taken by the support. One pin is then removed (one "constraint" is "relaxed") and the joint is displaced a certain distance, so that the frame takes the load (or most of it) which was being borne at that point by the support; the joint is then repinned to the support. The load at this joint has been "liquidated", leaving at most a small "residual". The success of the method lies in the fact that the forces in the members meeting in the joint which is relaxed, due to any displacement of that joint, can be *directly* calculated, so that the effect of the relaxation upon the forces at this joint and its neighbours is readily ascertained. One by one the "constraints" are "relaxed", tally being kept of the "residuals". These are thus progressively reduced, and when they are all deemed sufficiently small, the process is stopped. In engineering problems there is always uncertainty in the data; to speak of an "exact" solution is meaningless. All that can logically be justified—and indeed all that practical considerations demand—is a solution to within prescribed limits of tolerance. Within such limits, whose magnitude is clearly visible, such a solution is demonstrably obtained by the method. The limits of tolerance *could* be indefinitely reduced, were the process continued.

* The writer believes he has seen this affirmed, but can unhappily recall neither reference nor the exact phrase.

That is the simple fundamental idea, as applied to the original problem. But the idea alone will not suffice for the primary requirements of design calculation, nor will it explain the evolution by which an ever-widening range of problems is being brought within the scope of the method. To be useful, the idea must result in a technique neither unduly laborious nor unduly complex. In the above problem for a specific frame under any loading, the relaxation process uses two tables which are peculiar to the structure, and independent of the loading. The first is a table of " influence coefficients ", namely of the forces in members meeting at a joint, due to unit component displacement of that joint. These depend upon the geometry of the structure and upon the dimensions and elastic properties of the members. With the aid of these coefficients an " operations table " is next constructed. This records first the components of the resultant force per unit displacement acting upon any joint and upon its (undisplaced) neighbours. The results of this straightforward calculation are then converted into displacement and force components due to unit component force at the displaced joint. The entries in the operations table enable the displacement which will liquidate a residual, and the effect of this displacement upon neighbouring residuals, to be determined. The residuals are thus systematically liquidated, and after a number of operations will all lie within the limits of tolerance. That the process does in fact converge, in the mathematical sense, is deduced from the fact that the unique solution corresponds to minimum strain energy. The *rapidity* of convergence—much more vital from the computer's point of view—depends on the structure. But fortune favours the brave! Broadly speaking, the more complex and redundant the structure, the more rapid (relatively) is the convergence ; the method is more useful just where it is most needed.

This technique having been established, it is fascinating to see how one class of problem after another is brought within its scope. It is applied successively to rigidly jointed frames, to the distribution of errors in a surveying problem, to the distribution of current in an electrical network, and so to " gyrostatic " systems, and to normal modes and frequencies of vibration of systems of finite freedom ; even to continuous systems, *e.g.* to deflections, critical thrust, and vibrations of beams. In all cases the underlying theory is given, and " orthodox " methods of attack are indicated ; in connection with the illustrative examples worked out in detail, results obtained by exact analysis, with which the " relaxation " solution is compared, are given wherever they exist. Other known methods of approximate solution are also given, in more or less detail.

Gyrostatic systems present the first hurdle ; convergence of the relaxation process is no longer certain. But the Kirchhoff equations for electrical networks are known to be equivalent to those expressing minimum heat loss. This replacement is equivalent to " normalising " (in the statistical sense) the gyrostatic equations, and the existence of the minimum ensures convergence.

The minimum property contained in Rayleigh's principle is effectively used in solving the vibration problem. Here the process alternates between a more accurate determination of the frequency and a more accurate determination of the amplitude ratios which characterise the mode. The Rayleigh principle gives at any stage an upper limit to the frequency ; Southwell shows how a lower limit can also be obtained. By use of the orthogonal relations—suppressing the resonance at the lower frequencies when found—the higher modes and frequencies can also be found.

It is evidently essential in the relaxation method that the number of unknowns should be finite (even if large)—it is humanly impossible to relax,

one by one, an infinity of joints! Some artifice must therefore be used to bring continuous systems within the scope of the method. Four ways are given. In two the deflection at a finite number of points only is determined, and either the governing differential equation is replaced by the finite difference approximation, or approximate quadrature formulae are used in the corresponding integral relations. In the others the deflexion is approximated to as the sum of multiples of a set of suitable functions which severally satisfy the boundary conditions; two alternative methods ("relaxation by inspection" and "systematic relaxation") are given whereby the coefficients are determined.

The concluding chapter indicates the possibilities of the method as applied to non-linear problems, by brief description of work on the suspension bridge with an elastic truss, and on the torsion of I-section beams when plastic flow occurs. These problems are beyond the scope of exact analysis, at present. The second is a two-dimensional problem, and it is in the region of two-dimensional problems (and it so happens, of problems of physical rather than engineering interest) that developments at the hands of Professor Southwell and his research team are now taking place.

These relaxation methods, like all numerical ones, solve only *specific*, and not general, problems, although reduction to canonical form and the use of non-dimensional variables can greatly increase the generality of a particular solution. Since the engineer's problems *are* all specific, this is not a disadvantage. But the method has certain definite and considerable advantages over methods of exact analysis—as mentioned in the last paragraph, for instance. Indeed, in dealing with beams, non-uniformity of section makes negligible difference to the technique or labour of the relaxation methods, whereas each law of variation of the section leads to a different, specific, differential equation, and generally removes the formal solution outside the range of tabulated (of not even of "known") functions.

The kernel of the book is the set of LVIII numerical tables. According to temperament these will be regarded as fascinating or forbidding. But despite the care, generosity, and patience of Professor Southwell's textual explanations, the essence of the method cannot really be grasped without mastering the construction of these tables—as a colleague put it, "you must *also* do the 'homework'"! Only by solving similar problems in detail will it be believed that the labour involved is *not* excessive, and realised how simple and direct are the steps.

It is also to be emphasised that the method yields configurations, as well as forces and frequencies; in this it is superior to most other approximate methods, especially to those which attempt to obtain "characteristic numbers" only. The superiority is—in consequence—notable as regards the determination of the higher frequencies (and modes) of vibration.

The ideas and technique which this book describes, explains, and illustrates, make a notable contribution to our powers of attacking a wide variety of practical problems. We hope that it may not be too long before the inevitable sequel dealing with the present line of development can be written.

Finally, we must congratulate the author-printer collaboration upon the very effective use of a number of founts of type, and upon the pleasing appearance of the printed page and of the displayed mathematical formulae. Only in the numerical tables can any improvement be suggested, but here the use of old face ("heads and tails") numerals would have made them both more easily legible and less dull in appearance.

The price, also, is remarkable!

W. G. B.

A Bibliography of Orthogonal Polynomials. Compiled by J. A. SHOHAT, E. HILLE and J. L. WALSH. Pp. ix, 204. $3.00. 1940. Bulletin of the National Research Council, No. 103. (Washington, D.C., U.S.A.)

In recent years, there has been a great deal of progress in the field of orthogonal polynomials, a subject of great importance in many branches of pure and applied mathematics, in classical mathematical physics, in quantum mechanics, in probability and statistics, in the theory of interpolation and numerical integration, and so on. Any worker in the field is faced with several difficulties, the wideness of the subject, the diversity of its applications, and, above all, its extensive literature. Accordingly, in 1933, the National Research Council appointed a committee, consisting of E. Hille, J. A. Shohat and J. L. Walsh, to compile a bibliography on orthogonal polynomials. The result of the seven years' work is the excellent book now under review.

The bibliography is, apart from an omission to which I shall refer later, complete up to January 1938, and also takes into account the parts of periodicals available up to January 1939. The committee has endeavoured to give more than a mere list of titles; by an ingenious code they are able to give in a brief space a statement of the contents of each paper to which reference is made.

For example, one finds the following reference:

WATSON, G. N.

1. The harmonic functions associated with the parabolic cylinder.

[239] (2) 8 (1910) 393-421.

H : $\alpha b4 - b7 - b10 - b12 - b16 - e5 - f - g - i - i2 - m$ 1.1, $\beta a17$-$b14$, $ka3$.

From this one learns that the first paper on orthogonal polynomials written by Professor Watson was on "The harmonic functions associated with the parabolic cylinder", which appeared in *Proc. London Math. Soc.* (Series 2), Volume 8 (1910), pp. 393-421, and that in this paper the following topics are discussed:

Hermite polynomials expressed as nth derivatives, as simple definite integrals, explicitly and as hypergeometric functions.
Relations between Hermite polynomials in the real and complex domains.
Their relation to Tchebycheff functions of the second kind.
Their recurrence formulae.
The associated Christoffel-Darboux formula.
Their differential and differential-difference equations.
Their asymptotic properties inside the domain of orthogonality.
The expansion of analytic functions as series of Hermite polynomials, and the convergence of such expansions.
The application to boundary-value problems in potential theory.

The code adopted conveys a marvellous amount of information in a very brief space. My only criticism is that it would have been better to have bound the code separately to avoid constant turning of pages; for the code is not one to be remembered.

Unfortunately I cannot give the book my whole-hearted approval. In the first place, the authors have completely neglected all papers published in periodicals devoted mainly to mathematical physics, such as the *Annalen der Physik* and the *Physical Review*, in which interesting results in the theory of orthogonal polynomials frequently occur incidentally. Again, no reference whatsoever is made to two books which I personally find to be most valuable sources of information on orthogonal polynomials, viz. Professor H. Bateman's

Electrical and Optical Wave-Motion and his *Partial Differential Equations of Mathematical Physics*. This is a most surprising omission, especially when one finds that " Courant-Hilbert " is in the bibliography.

A second criticism is that the topical index is too abbreviated. For admittedly the bibliography is quite adequate if one knows, say, that Dr. F. J. W. Whipple proved somewhere or other a symmetrical relation between Legendre's functions with parameters $\cosh \alpha$ and $\coth \alpha$; but to find the reference if one did not know the author's name would be almost impossible—it would involve looking up and decoding some hundreds of references (covering a page and a half of the topical index) under the general heading *Legendre polynomials*.

Professor Shohat and his colleagues will, however, claim, and with some justification, that I am expecting too much of a bibliography, and that, within its limits, it is likely to prove invaluable to anyone who has not easy access to a large mathematical library; and with this, I agree.

As one would expect of a publication of the National Research Council, the book is extremely well arranged and beautifully printed. The only misprint I noticed was that Professor Milne-Thomson's name was spelled with a *p*.

E. T. C.

Introduction to algebraic theories. By A. Adrian Albert. Pp. viii, 137. 10s. 6d. 1941. (University of Chicago Press; Cambridge University Press)

Professor Albert is " fully aware of the serious gap in mode of thought between the intuitive treatment of algebraic theory . . . and the rigorous abstract treatment of the *Modern Higher Algebra* as well as the pedagogical difficulty which is a consequence " and the present work attempts to bridge that gap. The result is a work of great interest and charm which will be read with pleasure by many besides the budding specialists for whom it is primarily intended. Many of the concepts of the higher algebra are not difficult to grasp once we are presented with instances of them, and the concepts indeed have their origin in such instances. Some knowledge of the latter is, if not an essential, at any rate a highly desirable preliminary to higher studies. Here they are illustrated by a critical consideration of polynomials, matrices, linear spaces and polynomials with metric coefficients. The final chapter describes briefly the main objects of study in the author's standard work and introduces formally the group, ring, field, integral domain and ideal. For these conceptions the reader's mind is well prepared, though some rearrangement of the sections on ideals and residue classes might elucidate what seems at first sight a somewhat arbitrary definition. The topics of the earlier chapters are fully, though concisely, treated and well illustrate the power and elegance of the methods employed. For this very reason we have some little doubt whether the book quite fulfils its intended purpose. Its special merits will certainly be more keenly appreciated if the reader already has acquaintance with a more extended treatment of matrices and linear forms. As a summary and commentary, however, the work is wholly admirable and, clearly, important.

J. L. B.

Algebra. A text-book of determinants, matrices and algebraic forms. By W. L. Ferrar. Pp. vii, 202. 12s. 6d. 1941. (Oxford)

This book meets the need, acutely felt, for an algebra suitable to university students who, not desiring to be specialists, still require for their more serious pursuits some knowledge of the tools which algebra supplies and some acquaintance with its modern developments. Its perusal too will be of use

to those who propose to undertake later that most abstract of studies, the modern higher algebra, for it will familiarise them with some of the ideas whence that study has sprung and enable them to interpret their abstractions in terms of the concrete. Though intended primarily for the undergraduate the book will be a suitable and useful addition to the school library.

In a work with a limited aim some selection of material was necessary: many excellent textbooks, ancient and modern, deal very adequately with the theory of equations and elimination and Mr. Ferrar has ignored these topics. With perhaps less justification, though we appreciate the reasons he gives, he has omitted any mention of group theory and some might wish that he had sacrificed to this purpose the final chapter on invariants and covariants which is admittedly a sketch and stands apart from the remainder of the book. Those, however, who remember Elliott with affection and gratitude will rejoice in this evidence that in the university he served his studies are still pursued. Apart from this the book is divided into three portions of approximately equal length which deal respectively with determinants, matrices and linear and quadratic forms.

The definition of a determinant and the deduction of its characteristic properties abound in instances of the things which are easy to see being difficult to express. Mr. Ferrar avoids some of these difficulties by defining a determinant as an expression having certain properties and proving the existence and uniqueness of the expression. The result is a curious and interesting blend of the permutation and inductive methods. We note with pleasure a special section on the interesting skew-symmetric determinants and other types receive treatment in the text or examples. The treatment of matrices is, within the limits set, full, careful and lucid: it terminates with the study of latent roots and the proof that a square matrix satisfies its canonical equation. Equivalent matrices and elementary transformations are defined but the study of canonical forms is beyond the scope of the book.

The weapons thus prepared are applied to the study of linear and quadratic forms. Here Mr. Ferrar blends carefully and sometimes compares the explicit, though somewhat lengthy, treatment and the abbreviated symbolic methods. Greater use of the latter would have saved space but at the expense of intelligibility to those he has in mind. The method of reduction of quadratic forms is purely algebraic: does any English textbook contain an account of the "Method of the maximum", powerfully applied to this problem by Hilbert and Courant (*Methoden der mathematischen Physik*, I, p. 11)?

The examples are throughout well chosen to illustrate and supplement the text and the book bears many marks of the careful teacher: it will certainly be useful. It is perhaps unnecessary to add that where "display" means so much the Press well maintains its reputation. J. L. B.

New first course in the theory of equations. By L. E. Dickson. Pp. ix, 185. 10s. 6d. 1939. (John Wiley, New York; Chapman and Hall)

Although this book bears the title and is written in the spirit of a former work by the same author it contains a good deal of new matter. The style is less compressed, the examples are more numerous and the matrix notation clarifies the discussion of linear equations.

A scholar's presentation of the rudiments of his subject is always interesting: here it is lucid and truly elementary with a personal touch which prevents the book from being just one more addition to the literature of a hackneyed theme. The special attention given to the conditions for integral and rational roots is justified by an early chapter in which the author, inspired apparently

by a desire to lighten his post-bag, demonstrates the impossibility of trisecting any angle and enumerates the angles which can be trisected. The treatment of the roots of unity is applied to problems of geometrical construction and the short chapter on elimination contains an interesting application to the finding of imaginary roots. Among the symmetric functions discussed are many which are rational but not integral. As is now usual the formal treatment of the cubic and quartic equations is brief and the methods for the location of roots give little scope for originality of treatment.

The only methods of approximate solution discussed are those of Horner and Newton. In the exposition of the former the usual multiplication by 10 at each stage is omitted: this concentrates attention upon the essentials of the method but would seem to have practical inconveniences. The conditions for the success of Newton's method are fully illustrated by diagrams and the method is applied, both in text and examples, to non-algebraic equations. There is an admirable elementary treatment of determinants and matrices leading to the study of systems of linear equations. The treatment of these is luminous and particularly valuable. An appendix gives proofs of the fundamental theorem on symmetric functions and the fundamental theorem of algebra.

Little space is given to the fundamental concepts of "complex number" and "continuity". The treatment of the latter is based on intuition, and i is defined as one of the (postulated) roots of $x^2 = -1$; indeed, the discussion of complex numbers as such occupies but two pages. Is this an anachronism or a portent?

Students frequently complain that they understand the bookwork but cannot do the examples. Readers of this work will have no such difficulty: few of the exercises demand much ingenuity and there is a tendency, well illustrated on pp. 52-53 where one hundred numerical quartic equations are given for solution, to multiply routine examples at the expense of space which might have been utilised to indicate developments or test understanding of principles. We must, however, recognise the success which may attend the transatlantic method of making the path to higher studies as simple as possible. In this book that method is at its best. J. L. B.

The Foundations of Geometry. By G. DE B. ROBINSON. Pp. xi, 167. 1940. 9s. Mathematical expositions, No. 1. (University of Toronto Press; Humphrey Milford.)

This little book is an account of the axiomatic foundations of projective and euclidean geometry, written in a most attractive manner; it should appeal to those who are disposed to consider this subject difficult or obscure. The portion dealing with projective geometry develops the theory of projectivities and shows that the assumption of Pappus' Theorem leads to a proof of the fundamental theorem of projective geometry, and then explains how a coordinate system can be introduced on a purely geometrical basis. There are several accounts of this theory in existence, but I have read very few which are so clear and easy to follow as Professor Robinson's. At one or two points some of the details of the proofs are left to the reader, but to compensate for this the author throughout lays emphasis on the significance of his procedure. By this means the reasons underlying the various steps in the development of the theory are made clear, and the reader is not left at the end with the confused feeling that something has been produced out of nothing.

The rest of the book deals with the axioms of order and continuity, the foundations of euclidean geometry, the introduction of imaginary elements,

collineations and correlations. Here too the author succeeds in combining rigour of treatment with simplicity and elegance in exposition. A useful bibliography appears at the end.

This work deserves to be widely read; it should dispel some at least of the mysticism which still seems to be associated with the subject by some mathematicians. J. A. TODD.

Four-figure tables with mathematical formulae. By H. C. PLUMMER. Pp. iv, 89. 3s. 6d. 1941. (Macmillan)

This book is made up of two separate parts bound together into one volume. The first part consists of the familiar four-figure tables of Castle with certain modifications. The most interesting of these are the inclusion of an extra table to overcome the difficulty of the large mean differences in the early part of the table of logarithms, and the omission of tables of antilogarithms and square roots. Positive characteristics are used for logarithms of numbers less than unity.

The second part, which is a collection of formulae, occupies over forty pages and appears to cover every conceivable branch of elementary mathematics. There are in addition three pages of constants, including a large number of special interest to astronomers and physicists. B. M. B.

Differential and Integral Calculus. By Ross R. MIDDLEMISS. Pp. x, 416. 17s. 6d. 1940. (McGraw Hill, New York and London)

It is always interesting to read a good textbook published outside this country, not only for its own sake but also for that of the sidelights thrown on everyday life elsewhere. To quote a rather simple case from this (American) publication; a problem in the chapter on maxima and minima indicates that in the U.S.A. postal regulations allow total length and girth of parcels to be 100 inches.

On the other hand, the presence of such familiar themes ("the greatest permissible parcel") suggests that the book may be somewhat of a stock type. This one certainly is, though no condemnation is thereby implied. Only the very "usual" portions of the Calculus are treated, but the whole is well done, and the book is beautifully printed, even the smallest details being clearly and attractively presented.

Limits, elementary differentiation (and applications), and integration, followed by mean values, first and second moments and chapters on infinite series and Taylor's Theorem comprise the bulk of the contents. The last chapter gives a very brief introduction to Differential Equations.

The remainder deals with Partial Derivatives and Multiple Integrals. These two portions have been selected for especial comment as they typify the general approach. Magnificently clear diagrams illustrate, *e.g.* the tangent plane, or the interpretation of the element of volume found in triple integration, while the worked examples are displayed in such a fashion that there can be no possibility of confusion.

There are the customary numerical tables, a table of integrals (some 67 of them), a good index and a fairly complete set of answers to the collection of not-difficult examples. No error in questions or answers has been found after considerable testing.

Two small points do call for criticism. The hyperbolic functions are not mentioned save in a brief paragraph near the end of the book; their help in (amongst other things) integration is therefore lost. Secondly, the author appears to have a leaning towards the use of proper names. We find, for

instance,

$$\text{“}\lim_{x\to a}\frac{g(x)}{h(x)}=\lim_{x\to a}\frac{g'(x)}{h'(x)}\text{”}$$

stated as L'Hôpital's Theorem. This practice is useful in that it gives the learner a handle to attach to a piece of work and so provide himself with an additional aid towards its recollection. It is all the more surprising then to be unable to find the names of Newton and Leibnitz mentioned in any connection whatever in the whole of over 400 pages.

These are small blemishes; perhaps they seem more conspicuous since the book is, of its type, so good; but they might well be removed in future editions.

There is one more point in connection with the differing customs of different countries—this time one of mathematical printing. This volume uses throughout the period as a decimal point (i.e. 9.8 represents nine and eight-tenths) and the raised stop as indicative of multiplication (so that 9·8 means 72). Is there any possibility of establishing a "standard form"; and if so, could its use be made general? F. W. K.

The Factors of the Mind. An Introduction to Factor-Analysis in Psychology. By CYRIL BURT. Pp. xiv, 509. 21s. 1940. (University of London Press)

Readers of the *Gazette* may ask what interest a treatise on Psychology can be expected to have for them. The answer is that factor analysis, a subject whose origin was in problems of educational and vocational guidance and in metaphysical theories, and later took a statistical form, has now developed into a field of application of matrices. It may even lead to new discoveries in this field. Professor Burt found it necessary to investigate the latent roots of certain compound matrices, and obtained a result which appeared to be new, although it was afterwards found that it had been obtained long ago by Sylvester. Strangely enough, both psychologists and quantum physicists have been working with matrices on similar lines, until recently in complete ignorance of each others' work. The reason for this identity of method in two such very different subjects may, in Professor Burt's opinion, be ultimately traced to a common dependence on some more abstract branch of mathematics such as the theory of groups. Factor analysis may also be linked up with the statistical methods associated with the name of Professor R. A. Fisher, in particular the analysis of variance. We shall now give some account of factor analysis and Professor Burt's book, from the standpoint of a mathematician. This will, of course, be a one-sided view. The book was really addressed to students of psychology and education, but these and others interested in the philosophical method of approach adopted by the author should refer to reviews in psychological journals.

If the boys in a certain form at a school are examined in all the subjects of their curriculum, it is usual to estimate a boy's general ability by his average mark. However, two boys with the same average mark may have obtained it in different ways. One may have scored more highly in a group of scientific subjects and the other in a literary group. Even if two boys get the same total marks for the literary group, one may have appeared weaker in one specific subject, say English History, than in the other subjects of the group. This weakness may, of course, have been due to some accidental cause, such as antipathy between the boy and the teacher, or to illness in the examination. Thus in discussing the results, we may attribute them to causes or "factors" (*not* in the mathematical sense), which may be either

general, or *group*, or *specific*, or *accidental*. To put the matter more definitely, we try to express a boy's mark as a simple function, in particular a linear function, of certain other numbers which serve as his mental specification. Thus Smith's marks x in English History may be expressible in the form

$$x = ag + bv + cs + de,$$

where g, v, s, e are respectively Smith's general factor, verbal group factor, specific factor for English History, and error or accidental factor. a, b, c, d are called the loadings. They are taken to be independent of the particular boy Smith, and to depend only on the relation between the nature of the examination in English History and the abilities which it tests. If we have n examinations and p persons, the marks may be arranged in a rectangular matrix of n rows and p columns. In psychological work the subject of analysis is usually not school examination marks, but the scores in mental tests which are intended to test intelligence rather than acquired knowledge. It is convenient to reduce the scores in each test by their average in that test, so as to make their sum zero. If these reduced marks in a test are then divided by the square root of the sum of their squares, the results are said to be normalised. The product of a matrix of normalised scores by the matrix which is its transpose is a square matrix of n rows and columns, called the matrix of correlations between the n tests. There are two experimental facts concerning this matrix. The first is that most of its elements are positive, showing that a person talented in one direction is more likely than not to be talented in others. The second fact is that if the diagonal elements, which are necessarily unity, are suitably replaced, and small corrections are made in the other elements, the rank of the resulting matrix is low. If the rank is unity, the elements of every column are in the same ratio. This result is known as Burt's equation, or Spearman's tetrad equation, or hierarchical order, and Spearman deduced from it his famous two-factor theory. This states that the score of each test can be expressed as the sum of two parts, one a positive multiple of the general factor g, the other a positive multiple of a specific factor, independent of g and of the specific factors associated with the other tests. Thurstone considered the case when the rank r was higher than unity, and obtained r general factors, in addition to specific factors. A disadvantage of both Spearman's and Thurstone's factors is that they cannot be calculated exactly, but merely estimated. Hotelling has devised a method which gets over this difficulty, but has the disadvantage that his "principal components" (his name for factors) change in value whenever a new test is added. The number of these components is not kept low, but is equal to the number of the tests. However, in practice all but a few can be neglected. Burt himself has contributed to several of the older methods, but his latest work is on rather different lines. As before we start with a matrix of scores, and then reduce those in each row by the average of that row, so that the sum of the reduced scores in each row is zero. What is new is that by a suitable choice of units we contrive to make the sum of each column also zero. This introduces a symmetry between the relation of the matrix to the n tests and the p persons. If we post-multiply this matrix by its transpose we get a square matrix of n rows and columns whose elements are the unaveraged variances and covariances for the tests. Similarly if we pre-multiply, we get the corresponding results for the persons. By the Sylvester-Burt theorem the non-zero latent roots of these two product matrices are the same. Burt's final results consist of general and group factors, which can be calculated exactly. There is no indeterminacy, a great weakness of some of the older methods, and Thurstone's laborious "rotation of the axes" is not needed. There are no specific factors. The subject is still in the realm of controversy,

but it may be said provisionally that in several respects Burt's methods appear the most promising yet found. Of course there are some who deride the whole tribe of factor analysts. In the *Harvard Educ. Review* (May 1939, p. 287) E. E. Cureton says, "Factor theory may be defined as a mathematical rationalization. A factor analyst is an individual with a peculiar obsession regarding the nature of mental ability or personality. By the application of higher mathematics to wishful thinking, he always proves that his original fixed idea . . . was right. . . . He usually proves that all other factor-analysts are dangerously insane . . . and searches about for some branch of mathematics which none of them is likely to have studied in order to prove that their incurability is not only necessary but also sufficient."

The mathematician who is not also a psychologist may leave it to others to dispute over the interpretation of the work, and be content himself with the interesting occurrence of matrix and even quantum theory in an unexpected place. But he should be warned that Professor Burt, who is a clinical psychologist, philosopher, logician, Greek scholar, and physiologist, as well as a mathematician, has not written his book for those whose interests are limited to mathematics alone. He considers it desirable in the interests of the ordinary reader to approach the subject from the philosophical point of view, though he admits that "to the advanced mathematical psychologist, this mode of presentation may seem alike prolix and inconclusive. To forestall this criticism, I had originally intended to append a systematic summary, giving rigorous algebraic proofs of the essential formulae. . . . But the recent publications of Thurstone and Thomson—both remarkable for the lucidity and thoroughness with which they have covered these more technical aspects—have rendered my own attempt not only superfluous but largely out-of-date. Accordingly, I have merely retained an Appendix on working methods which I hope may be of service to the practical investigator, not only in psychology, but also in many other sciences—medicine, agriculture, biology, economics, and the various branches of social science—where, as I believe, factor-analysis might often be applied with ease and advantage." H. T. H. PIAGGIO.

Statistical procedures and their mathematical bases. By C. C. PETERS and W. R. VAN VOORHIS. Pp. xiii, 516. 31s. 6d. 1940. (McGraw-Hill)

The authors, two members of the staff of the Pennsylvania State College, say that the book might better be called "Some Statistical Procedures and a Little Insight into the Mathematical Bases of a Few of Them". It is a first edition but is a revision of a book privately published in 1935: there are at least half-a-dozen references in it to matter that is of interest but can be found only by using the earlier book. The Procedures relate primarily to educational research and practically speaking examples are drawn from no other field of study. It is a comprehensive volume, and almost everything that is likely to be required by the psychologist-statistician is here. An indication of the chapter contents may be of value.

I. "A Little Calculus." This chapter is inserted "to prepare the reader who has not previously studied calculus to follow the derivations". Neither in this chapter, nor elsewhere in the book, would the criteria of rigour of an English text be satisfied, and it is not clear what mathematical knowledge the reader is supposed to have. Two pages are (later) used in obtaining Σn^2, though interpolation and point of inflection are used without definition or explanation.

II. "Measurement of Central Tendencies."

III. "Measurement of Variability." This includes the proof of Sheppard's Corrections. A non-educational example here involves a J-curve: corrections for abruptness are not used in calculating the constants of the distribution.

IV. "Rectilineal Correlation." The example given at the end of the chapter—correlation between size and *per capita* cost—would involve the spurious index correlation against which there is a warning in Chapter VII.

V. "Reliability of Statistics." This refers quite often to the normal curve, which is not dealt with till Chapter X, and to "Student's" t, which is only considered properly at p. 420, as well as to fiducial limits.

VI. "The Reliability of Differences." This includes Fisher's table for t. It is unfortunate that certain tables of tabulated functions are included in the text and are numbered serially in with the illustrative experimental tables: there is no index of tables so that it is difficult to hunt this one of Fisher's, for example, down to p. 173 in the absence of any specific reference.

VII. "Inferring Coefficients of Correlation for Changed Conditions." This chapter includes the Spearman-Brown formula for "stepping-up" reliability, the correction for attenuation and a rather full treatment of the effect on correlation of heterogeneity and selection; no reference is made to the fact that this theory is due primarily to Karl Pearson on Natural Selection.

VIII. "Partial and Multiple Correlation." In addition to the ordinary theory, the Doolittle method of computing is given fully. It would have been helpful if some guide about the number of significant figures to be used and retained had been given, especially in view of the authors' note of the effect of small differences in the raw r's.

IX. "Multiple-Factor Analysis." After an outline of Spearman's Tetrad Differences, Thurstone's Central Method is dealt with. This includes the two steps regarded by Burt as fallacious—the use of the highest inter-test correlation for the communality, and the rotation to maximise zero loadings (and thus make all weightings positive and no factors bi-polar)—and involves the sign-changing wangling after removing the first loadings.

X. "The Normal Probability Curve." This has been rather long delayed. The graph (Fig. 19) for the histogram should state the value of n that has been used.

XI. "The Correlation Ratio." In addition to η the theory of what the authors (after Kelly) call the epsilon technique is developed.

XII. "Analysis of Variance." An outline of Fisher's method is given here with an explanation of Latin Square (the definition being in two parts, half a page apart), Degrees of Freedom, etc. The illustration given is of milk yield, and the authors definitely prefer for educational research their ϵ to "the now popular analysis of variance". Snedecor's F is defined and is preferred to Fisher's z.

XIII. "Further Methods of Correlation." This chapter deals with Biserial, Tetrachoric and Mean-square-contingency correlations, and includes the authors' work on what they call widespread classes (that is, using only the two extreme tails of a distribution).

XIV. "Chi-Square." Pearson's χ^2 is here considered, together with t, and an outline of Huntingdon's generalisation. There is a further reference to degrees of freedom, coming down on the side of K.P. rather than Fisher in its enumeration. Table XXXIX here is inconsistent in column f_1 with the sixth column of Table XXV of p. 306, whilst the calculations here are surely too drastically rounded.

XV. "Curve Fitting." Again rather a delayed topic. In addition to the Pearsonian method of fitting by moments (the authors do not seem to realise the relationship between the several usages of the term "moments", partly possibly because of their vagueness about the difference between ordinates and areas) there is something about normal equations (here defined

but used in Chapter VIII) the Logistic Curve (no reference here to Yule—only to Pearl) and the Gompertz law.

XVI. "The Technique of Controlled Experimentation." Here the book comes out even more strongly than before in favour of, *e.g.* matching groups as against the Fisher technique of replication. The authors' lack of rigour, about degrees of accuracy, in not defining what is experimentally "the same", "alike", "a few points" (*cf.* p. 448) here, we suggest, has led them astray. There is nowhere in the book any definite lead given to the size of the populations (or samples) for which the formulae are appropriate, save on p. 456 which suggests that $N=36$ gives a "large" sample.

Appendix. This consists of two normal curve tables, from *Tables for Statisticians and Biometricians*, "oriented" in terms of q and of x/σ, two of "Student's" t from his *Metron* paper, a table of the authors of ϵ^2, Elderton's P table for χ^2, two tables for use in connection with tetrachorics, and a table for ranks (percentiles) for various values of correlation coefficients.

The book as a whole is therefore an ambitious and valuable survey of the field. There is nothing about Efficient Statistics, nothing about the Q technique of correlating persons instead of tests. But in the main everything that is commonly needed in the field with which the authors are concerned is here. Our main outstanding criticisms are on relatively minor points. We have already indicated that the mathematical outlook is insufficiently rigorous: a statement (p. 264) that the residuals are so small that they are obviously due to chance is unconvincing. The system of references is not always satisfactory. Tabulated functions appear in casual tables in the text: equations and formulae have reference numbers at either end of the line and reference to them is difficult when page and index references are not given (is, for example, the F of p. 352 Snedecor's F or formula F of higher up?): page references are not always accurate: memoir references do not always give initials (for example, Jones, p. 128; Pearson, p. 159; Soper, p. 365) nor date, whilst there is some confusion between *Phil. Trans.* and *Proc. Royal Society*. The order of presentation, though possibly convenient pedagogically, is awkward: in at least a dozen cases there is used a symbol, term, result or table that is only properly dealt with (if at all) later. Thus Σ is used on p. 12 and defined on p. 44. The authors take rather an indefensible line in suggesting (p. 358) that statistical procedures are not usually conveniently adapted from one field of study for another. Yet they claim (pp. 122, 280) on grounds of genetic theory that mental traits are normally distributed. Mental traits are unmeasurable: we can only determine them indirectly by some more or less subjective measures of certain manifestations. And it is as reasonable to claim that the traits behind the measures follow the same frequency law as the measures as it is to claim that the stature of a man, determining his bulk and thus his weight, follows the same frequency law as does his weight, or his business capacities as the income earned by these.

The book is nicely got up. There are a few misprints: it is not clear (apart from the context) what is meant by $dy/dv \,.\, dv/dx$, and on p. 436 an index letter is dropped in $y=ki^{rt}$. There are some usages that read strangely to us. We recognise "componendo" in "composition" but are not prepared for a "complex value" to mean a "complicated expression" nor "percentages" (pp. 437, 440) "fractions". And we get tired at the ninth appearance, on the last page of the text, of the idea expressed on the first page of the preface, of mathematical and statistical tools being "magic". But these are minor points, and should not prevent any serious difficulty to the British reader. He should find the volume of great assistance if he is working in the field the authors have surveyed so carefully. FRANK SANDON.

General Mathematics. I. By L. TURNER. Pp. viii, 238. 3s. 6d. 1941. (Arnold)

This book is written by the head of the mathematics department at Coventry Technical College and has been arranged and developed in accordance with recommendations given in *Suggestions in regard to teaching in Junior Technical Schools* (B. of E. educational pamphlet, No. 113, 1937) and the *Spens Report* (1938).

In technical schools mathematics must be regarded primarily as a technical subject and the pamphlet requires mathematical teaching in these schools to give: (1) a knowledge of ordinary arithmetic, especially mensuration; (2) ability to handle formulae confidently, to be able to transform them, to substitute in them and to evaluate them; (3) ability to handle logarithmic tables and tables of trigonometric functions quickly and with understanding; (4) ability to understand the use of equations and to solve simple and quadratic equations; (5) a thorough understanding of graphical methods of presentation and of the use of graphs to illustrate or to solve problems arising in science and technology; (6) a knowledge of the principal geometrical properties of rectilinear figures and the circle, and some acquaintance with the ellipse and parabola: the use of the locus is of special importance; (7) a knowledge of simple solid geometry; (8) a knowledge of the elements of trigonometry as far as the solution of triangles; (9) an introduction to the ideas and notation of the differential and integral calculus.

"Much that is included in the ordinary academic course must be excluded in order to concentrate upon matters of real value in subsequent practice." To those who would like to retain, for example, in algebra, multiplication and division of multinomials, exercises in fractions, ..., on the ground that these are necessary for technique, the pamphlet replies, "Technique is a means to an end; in this case to the proper understanding and treatment of science and engineering and is best acquired by continual practice in exercises and applications drawn from them."

"It must not be inferred from this that the mathematical syllabus is to be a thing of shreds and patches. It is the business of the teacher to develop the course so that it forms a coherent whole. The needs of other branches may involve departure from strict logical order but nothing should be treated entirely empirically or without some logical justification, and if the treatment has at any particular moment to be rather incomplete a return to fuller treatment may be made later. The task of the teacher is made much easier in this respect if the subject is treated as one without any subdivision into algebra, geometry, trigonometry, etc., and use is made of whatever method is appropriate to the needs of the problem in hand."

The pamphlet then gives the desirable content of the normal course, and Mr. Turner follows its recommendations, the first year's course being given in this book. The result is very interesting and instructive for the "academic" teacher. It is of course possible to find fault with the presentation in minor details and in general for going at too fast a pace, but on the whole the author has made a good job of what he set out to do. The book deserves to be widely used in technical schools. It should also prove very useful as a textbook for A.T.C. use.

It is less evident, however, that it will be found as suitable as the author claims for use in the lower forms of a Secondary (Grammar) school. Let us judge it here by the Spens Report.

The Spens Report recommends (*a*) that mathematical teaching should occupy less time in the school curriculum; (*b*) that it should be more directly linked with practical problems of life; (*c*) that the assumption that pupils

must understand the underlying principles of the new tool before they use it must be discarded; (*d*) that there should be no separation of arithmetic, algebra, . . . into water-tight compartments. So far, so good. The adoption of Mr. Turner's book would give effect to these recommendations for: (*a*) The one year's course covers the gist of what takes about two years to do (properly?) in the normal grammar school syllabus. The last two of the eight chapters are on Similarity (including the trigonometry of the right-angle triangle) and Mensuration (as far as the sphere). (*b*) There is plenty of utilitarian interest, chiefly technical and chiefly in the examples. (The "problem method" (Spens Report, p. 238) is however not used.) There is also some nice historical interest about numeration and π. (*c*) The use of logarithms is introduced in Ch. 3, Ch. 1 being about whole numbers, fractions, scales, percentages and simple statistical graphs, and Ch. 2 on decimals and the metric system. (*d*) There is no water-tight separation, yet there are no surprising amalgams. The ruler and geometrical bisection are used to good effect in fractions and many of the questions ask for dimensioned drawings to be made. Quadratic expressions and factors are introduced with the areas of rectangles. Trigonometry and graphical representation of proportion come in Ch. 7 on similarity. This is certainly not more than the ordinarily efficient academic teacher with his three or four separate textbooks does in barrier-breaking. The chances for barrier-breaking come usually in particular problems and probably it is not practicable to do much more than Mr. Turner has done in an elementary textbook. The Spens reporters seemed unduly worried about these alleged water-tight barriers. Perhaps they had seen the B. of E. school time-table returns on which, to please officialdom, we must specify every mathematical period as A, Alg., Geom., or T.

The most obvious criticism that can be made about the book is that it goes too fast. For instance, Ch. 4 contains all the algebra except quadratics. In this chapter of 32 pages we start with 2 pages on the use of symbols followed by an exercise of 25 questions, the last six on simple fractions. Then follow 4 pages on the addition, subtraction, multiplication and division of negative numbers and on brackets, and an exercise of 17 questions; then a short section with drill exercises on how to manipulate logarithms of numbers less than unity. Then the writing of formulae (one page) followed by 21 questions on generalised arithmetic, construction and interpretation of formulae. Then 3 pages plus 20 questions on simple equations: No. 1. $14x = -42$, No. 10. $3(5-6x)-5[x-5\{1-3(x-5)\}]=23$. Problems leading to simple equations, 2 pages plus 20 questions (none of any utilitarian interest). Changing the subject of a formula, 2 pages plus 23 examples, some difficult; for example, No. 20: $C=nE/(R+nr)$, find n. Further use of logarithms; $\log \sqrt[n]{a} = 1/n \log a$, and square and square root tables, $4\frac{1}{2}$ pages plus 20 questions. The chapter, like every other one, finishes with a revision exercise of 10 questions. Surely no boy or girl can be expected to digest this so quickly. Yet if Mr. Turner were to introduce more drill questions would he not be, in the words of the Spens Report, "Clogging it with exercises in pure technique of little practical interest or value"?

The same high degree of concentration is to be found in the next two chapters on Angles and The Triangle. These consist of a catalogue of definitions and geometrical facts, the acceptance of which is prepared by experimental investigations. The treatment of Pythagoras (which is considered sufficiently unobvious to warrant a proof) is excellent. The 3, 4, 5 triangle is drawn with the squares on the sides ruled into 9, 16 and 25 unit squares. An experimental dissection is given to the class to carry out individually and then comes Euclid's formal proof. The *raison d'être* of many strange

facts given before will now become apparent to the boy as being required for this proof. There is, however, very little deduction of general results in the course and children could not get much idea of proof or much practice in logical reasoning from it. Mr. Turner can claim in his support " Geometry is still dominated by the notion that it should be a deductive science " (Spens Report, p. 242) but his opponents can counter here with " Our witnesses were disposed to think that the reasoning capacities of children are rather underestimated by current methods of education. . . . Logical reasoning must not be regarded as a specific faculty but rather as a *technique* that can be taught. . . . Ideas of proof and systematic discovery and experiment are still to a great extent foreign to the ordinary pupil " (p. 128). I think that Mr. Turner has certainly cut out live as well as dead wood. His tree may bring forth utilitarian fruit more abundantly but it is not a very beautiful tree.

The book is well printed and contains clear diagrams. Answers are provided and a random selection of these proved free from error. Anyone who is interested in the need for the reform of the school mathematical syllabus should certainly examine this attempt to satisfy simultaneously the requirements of the technical and grammar schools. R. C. L.

Mathematico-Deductive Theory of Rote Learning. By C. L. HULL, C. I. HOYLAND, R. T. ROSS, M. HALL, D. T. PERKINS, F. B. FITCH. Pp. viii, 329. 21s. 6d. 1940. (Yale University Press ; Oxford University Press)

This publication of the Institute of Human Relations at Yale is the result of an interesting attempt at cooperation by psychologists, mathematicians and a logician. It is presented for consideration as an essay in method, quite as much as a contribution to the theory of certain types of learning. In the words of the authors, " it is judged that its chief value consists in the large scale pioneering demonstration of the logico-empirical methodology in the field of behavior."

By " rote learning " is understood, in the context of this study, the memorising of a string of nonsense syllables. In the standard procedure here used, the subject of the experiment sees each syllable of the series in succession, meanwhile pronouncing each aloud as they appear before him. At the second showing which follows at once, he now tries to forecast each syllable immediately before its presentation. The process is continued without intermission until the subject has memorised the entire series, as evidenced by two successful forecasts of the entire series. A detailed record is meanwhile being kept of the entire sequence of successes and failures. Such learning behaviour has been studied in much detail by experimental psychologists for over half a century. The conditions for the formulation of a theory may thus be considered ideal. Not only is the subject-matter available in a form which positively invites mathematical classification and arrangement ; there is also no lack of well-established empirical laws to explain.

The mathematical theory developed by Hull and his collaborators is largely suggested by experimental research on conditioned reflexes. Among the assumptions used are :

A. That each syllable presentation induces a *stimulus trace*, extending with diminishing intensity to the end of the presentation cycle.

B. That each syllable presentation induces a finite increment in an associated *excitatory potential*.

C. That at the instant that excitatory potentials are added, finite *inhibitory potentials* are added to the stimulus traces of other syllables.

The reader will notice that the " primitive notions " italicised in the last paragraph, are unobservables, functioning as symbolic aids to description within the theory, without attempt at independent definition in terms of neural or other physiological organisation.

Some fifty theorems and over one hundred corollaries are deduced from the postulates. It is notable that the authors do not claim exact verification of the consequences of the postulates. Indeed in one crucial case, that of Jost's Law, no attempt is made to conceal disagreement with " one of the most firmly established empirical principles in the field of rote learning ". But the candour with which such defects are admitted disarms criticism.

A cursory examination suggests that the mathematics involved, like so much applied mathematics, is rather dull and pedestrian. The noble attempt by Dr. Fitch to translate the whole proceedings into the language of symbolic logic may be of more permanent interest. He has at any rate done something to meet Poincaré's old reproach that symbolic logicians never use their symbols.

In view of the success with which the authors have achieved their aim of presenting a " concrete, large scale demonstration of what we mean when we speak of method in the social sciences ", this book deserves the attention of others than professional psychologists (who will read it as a matter of course). It should provide thought-provoking material for all interested in the philosophy of science. M. BLACK.

Contributions to the Fundamentals of Arithmetic. By MUKAND LAL DHAWAN. Pp. vi, 69. 7 annas. 1940. (R. S. Lala Mukand Lal Dhawan, Lahore)

This little book presents short methods for carrying out certain arithmetical operations. The methods show how long multiplication and division of number, recurring decimals of the order 97017/29999999 to 35 places, extractions of square and cube roots may be determined mentally. The author shows how these determinations may be made rapidly and without the former drudgery of extensive mechanical work.

In the foreword it is suggested that " the author brings a breath of fresh life to the dry bones of arithmetic and sets up a 'new order'". The book did bring a breath of fresh life to the subject for me, but I believe that about twenty years ago a Mr. Russell,* formerly Headmaster of the East Bristol Central School, gave a very similar treatment of the subject. Because of this similar work Mr. Russell was often referred to as the Mathemagician. The foreword goes on to say that, " This simplification of the first aspect of mental discipline we impose on our children may well be said to be a realisation of the dream of the World's Childhood ". Apart from the multiplication method, however, it seemed to me that in order to be able to apply the unique short methods suggested by the author more mature minds are required than are possessed by our pupils at the times we normally introduce these subjects. The application of these methods certainly demands at least a knowledge of negative number—a subject usually dealt with only in courses of work on algebra.

I found that pupils of eleven could readily apply—even if they could not understand—the author's method of performing multiplication. I am afraid, however, that the peculiarity of the method led them to regard it as being something in the nature of a party trick. From the point of view of interest I give the method here. The explanation of Mr. Mukand's method is my own. His is too elaborate and detailed for inclusion here.

* Perhaps some member will be able to furnish me with particulars of any of Mr. Russell's publications on this subject.

Example : 381951×21322.

Method :	3819510000	Reformed multiplicand
	22312	Inverted multiplier
	8143959222	

Invert the multiplier and place its original units' figure beneath the units' figure of the multiplicand. In the empty space above each figure of the inverted multiplier place a nought.

1. Multiply each figure above the multiplier by the number below. Thus $2 \times 0 = 0$, $1 \times 0 = 0$, $3 \times 0 = 0$, $2 \times 0 = 0$, $2 \times 1 = 2$. Put down 2 : nothing to carry.

2. Now with each figure of the multiplier multiply each figure of the multiplicand which is one place to the left of it and add together the resulting products. Thus $2 \times 0 = 0$, $1 \times 0 = 0$, $3 \times 0 = 0$, $2 \times 1 = 2$, $2 \times 5 = 10$, 10 and 2 are 12. Write down 2 and carry 1.

3. Repeating the process with multiplicand figures 2 places to the left of the multiplier figures we have : 0, 0, 3 and 1 carried $= 4$, and $10 = 14$, and $18 = 32$. 2 down and carry 3.

Other steps are :

4. $3 + 0 + 1 + 15 + 18 + 2 = \mathbf{39}$.

The addition proceeds as follows :

5. 3, 5, 10, 37, 39, **55**.

6. 5, 15, 24, 27, 43, **49**.

7. 4, 22, 23, 47, **53**.

8. 5, 7, 15, **24**.

9. 2, 18, **21**.

10. 2, **8**.

Stages 1, 2 and 3 can be simplified by using respectively only the first, the first two and the first three left-hand figures of the multiplier.

After a little practice I found that I quickly overcame the initial confusion of figure placing and that I could obtain the final product with a pleasing rapidity.

The unique nature of the work will be of interest to mathematicians generally. H. W.

Functional arithmetic through citizenship. IV. One and all. By J. TREVELYAN and J. MORLEY. Pp. vi, 90. 1s. 9d. 1940. (Longmans)

Previous books have discussed the arithmetic of home building, household budgeting, travelling, wages and insurance, and the last of the series deals with the financial organisation of the community. The theme covers the method of obtaining the community's money, and its allocation to various services, such as health, roads, schools. As in the other books, the information is put in a clear and simple, but not childish, way for children to read. Mental and ordinary arithmetic exercises are given, with suggestions for activities for the children themselves in obtaining personal and local material to enrich and supplement that given in the book. Finally, though no small item, there are delightful sketches, ranging from " Councils of the Cave Men " to " The Earnest Student ".

The arithmetical field covered is percentage, averages and graphs, with stress on intelligent use of large numbers, and the management of approximate results from actual data, as opposed to the obliging numbers of the mathematicians, numbers which all too often prove to have a common factor of 17.

As before, when I was able to say that an arithmetic book contained a moral, so, in this book, there is a valuable moral, forcefully demonstrated

though actually unexpressed. Here it is the stress on community *services*. All too often rates and taxes are accepted as an unrighteous extortion of a bogey "Government", with no suggestion that many services, far beyond the individual, are possible to the community through joint payments. In this book, money is collected from the community and money is used by community representatives for community purposes.

The aim of the authors is that these books should teach the processes of arithmetic, here percentages and graphs, through the material of everyday life. As in the previous reviews, I must express my dissent from this treatment, as far as maintaining that to teach percentages and discuss budgets together confuses both. To me this book is excellent in giving vivid significance to processes already acquired in "unreal" arithmetical examples. Given plenty of preliminary drill and practice examples in percentage, this last book completes an excellent set of books in citizenship for senior elementary, central or junior secondary school classes. These books would, I think, be valuable for older secondary pupils, particularly girls, who have despairingly concluded that the intricacies of mathematics are not for them. Such children would discover that in adult life you are given the rates at 11s. in the £, not expected to discover this in the last line of a problem. They would find, with renewed hope, that a firm grasp of a few very simple principles was the key to a better knowledge of a vast part of the background of life for an ordinary citizen. With these books, arithmetic would live again through citizenship.

I have been writing this review while fire-watching at my school, and have shown the book to my companion—who is not a mathematician. She was much taken by the method of presentation (and by the pictures) and remarked "I should not have thought that arithmetic could look so attractive. What a pity we did not have such a book in our schooldays. Perhaps the battle with our income-tax forms would be more successful."

Surely, I can finish on this recommendation. D. E. S.

The development of mathematics. By E. T. Bell. Pp. xiii, 583. 31s. 6d. 1940. (McGraw-Hill)

In recent years, Professor Bell has written a number of books about mathematics, suitable for those of us who would like to know more of the modern developments, even though we cannot hope to study them in the original memoirs of the current periodicals. This latest work may be regarded as complementary to *Men of Mathematics* (1937) but it does not seem to me to be quite so good a book. There are two main reasons for this judgment. One is that clearly the author must assume that much of the mathematics he discusses is known to his readers; he has made a considerable and often successful struggle against this handicap. The second reason is that the many comments are often irrelevant and sometimes misleading. Thus the frequent references to the "barbarism", "lack of sense", "bickering sects" of Europe, since they cannot be intended to have a strictly mathematical interpretation, must be regarded as irrelevant. We can enjoy the very sound remark that "the Heaviside tragi-comedy degenerated in three acts into broad farce: the Heaviside method was utter nonsense; it was right, and could be readily justified; everybody had known all about it long before Heaviside used it, and it was in fact almost a trivial commonplace of classical analysis." But then the contrast between "engineers . . . using Heaviside's methods about 1910" and Heaviside's "unsolicited mathematical rehabilitators" conferring "all the rights and privileges of mathematical legitimacy" is exaggerated by reference to "Bromwich in 1921". The interpretation by

contour integrals was given by Bromwich and Wagner, independently, in 1916; and certainly Bromwich had been working at the problem * for some years before 1916.

If the reader is prepared to accept many of Professor Bell's comments as flippancies; if he realises that Bell's tendency is to under-value rather than to over-rate; if he appreciates that Bell, justifiably irritated by the vague rhapsodies of uncritical (and often non-mathematical) minds, is keener to note the opportunity lost than the progress made; then he will find in the book much information, clearly conveyed and effectively arranged. It is hardly suitable for schoolboys; but the Sixth Form teacher should read it, and undergraduates would appreciate its assistance in coordinating their knowledge.

Of the 23 chapters, the first six take us up to the sixteenth century, and so deal mainly with the Greeks and Babylonians. The author is not so severe on the Greeks as he was in *Men of Mathematics*, though he does not share the view expressed by Professor G. H. Hardy in *A Mathematician's Apology* (p. 21): "The Greeks first spoke a language which modern mathematicians can understand; as Littlewood said to me once, they are not clever schoolboys or "scholarship candidates", but "Fellows of another college'."

The period 1637-1940 is looked upon as one of more or less steady development, though possessing some well-marked crises, such as the new emphasis on rigorous demonstration in the early years of the nineteenth century, marked by the work of Cauchy, Abel and Gauss, and the more recent probings into the logical foundations of mathematics begun by Frege, Whitehead and Russell, Hilbert and Brouwer. Chapters 15 and 20, "Contributions from geometry" and "Invariance", are excellent sketches of the flux of geometrical thought, from Lobachewsky and Bolyai, through Cayley and Gordan to Klein and Lie, and so to Poincaré and the topologists. Perhaps a mathematician writing about mathematics finds most difficulty when on his own ground; certainly the chapter on "Rational arithmetic after Fermat" seemed to me to be dull.

The last chapter "Uncertainties and probabilities" gives a vivid picture of the feverish, turbulent activity of mathematical philosophy today. The second part of the chapter discusses the statistical method; since elsewhere in the book the author has laid great stress on the axiomatic method in pure mathematics, he would seem to have missed a chance here of clearing up some of the confusion which has been caused by neglect of the axiomatic method in probability.

I hope I have made clear the fact that there is much sound information and much good entertainment to be had from this rather expensive but beautifully printed and neatly bound volume. T. A. A. B.

Barlow's tables of squares, cubes, square roots, cube roots and reciprocals of all integer numbers up to 12,500. 4th edition. Edited by L. J. COMRIE. Pp. xii, 258. 8s. 6d., postage 6d. 1941. (Spon)

The third (1930) edition of Barlow was the first to be revised by Dr. Comrie. The main tables then ran up to 10,000; in this new edition these have been extended up to 12,500. The book really contains a good deal more information than the title indicates, since we are given, among other things, a table of $\sqrt{(10n)}$ from 1,000 upwards, the fourth, fifth, ... tenth powers of n up to 100, $n!$ up to 100, the 11th to the 20th powers of n from 1 to 10, binomial coefficients, constants, and other useful matter. The tables for $\sqrt{n}$, $\sqrt{(10n)}$, $\sqrt[3]{n}$ and $1/n$ from 1,000 upwards are provided with interlinear first differences.

* See, for instance, Bromwich, *Gazette*, XIV, p. 227.

There is also a concise and masterly introduction on the arrangement and use of the tables.

Barlow's original edition appeared in 1814, the second edition, revised by De Morgan, in 1840. Dr. Comrie's innings makes the book appear set for a double century. T. A. A. B.

The Nineteen Forty Mental Measurements Yearbook. Edited by O. K. BUROS. Pp. xxiii, 674. 6 dollars; orders sent directly, discount 10%. 1941. (Mental Measurements Yearbook, 32, Lincoln Avenue, Highland Park, New Jersey)

For all who are interested in measurement—whether of intelligence, aptitude, or achievement—this book is well worth purchasing. The series of yearbooks of which it is the latest issue originated in a realisation in 1934 of the amazing number of tests being published each year and the paucity of information available about most of them. Material too often has been widely advertised by publishers for authors who have omitted to indicate the size or distribution of the population tested and the validity and reliability of the scores obtained. This yearbook, with its comparable predecessor of 1938, should do much to secure the suppression of such inadequately prepared compilations.

Two hundred and fifty reviewers have in this book cooperated in the criticism of a very large selection of old and new tests and books on testing ranging in date from 1915 to September 1940. Differing viewpoints are represented. Cross-references to other reviewers are most competent, and details of fact have been most accurately checked. (A detailed index of subjects would have increased still further the accessibility of the information supplied.) The usefulness of the book as a whole is very great, and its personal interest is considerable to readers with some knowledge of the publications of the contributory reviewers.

The sections on mathematics are of particular interest, especially to those who are sometimes tempted to minimise the contribution which standardised testing has made to the measurement of aptitude or achievement in geometry, trigonometry and algebra. C. M. F.

What are Moray House Tests? By G. H. THOMSON. Pp. 8. 6d. 1940. (University of London Press)

This is clearly written and informative, but somewhat disappointing. It is offered for sale and therefore purports to be a booklet; but much of its contents is on the level of a publisher's advertisement. As such it tells us (quite truly) that in the making of the tests all the best experimental and statistical methods have been used, that the tests are devised for the classification of pupils at about the age of eleven, that they are easily administered and unambiguous in their scoring, that their contents are kept secret and that they are in the main supplied only to Directors of Education.

Some enthusiasm creeps into the expression of the belief that individual happiness and the good of the community are both furthered by an education which is suited to ability; but the booklet drops again to the level of bare description in its brief reference to test construction, standardised scores, reliability, and its rather surprising acceptance of the terminology of "high saturation with the factor of general intelligence (g)".

An opportunity for the encouragement of professional growth seems to have been lost by the omission of a bibliography and an introduction to the very excellent published reports of the Thomson Research Group or the Scottish Council for Research in Education. C. M. F.

PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD.
THE UNIVERSITY PRESS, GLASGOW